3rd Year Higher Level Coordinate Geometry

For the lesson on
26/1/17
At Temple Carrig School, Greystones
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1. Title of the Lesson: "*Coordinate Geometry meets Synthetic Geometry*"

2. Brief description of the lesson

Students are presented with a quadrilateral, the four corners of which are points lying on a circle. They are asked to find as many different approaches as possible to show that the quadrilateral is a rectangle.

3. Aims of the Lesson:

***Long-range/thematic goals:* For students to**

- Find multiple approaches to solve problems
- Appreciate that mathematics can be used to communicate thinking effectively • Make connections between seemingly different topics they have studied.
- Become more creative when devising approaches and methods to solve problems
- Develop enthusiasm for the subject by engaging with a stimulating and challenging activity • Apply the concepts that they have already studied in unexpected contexts.
- Appreciate and develop their skills of deductive reasoning.
- Develop an appreciation that mathematics can be used to communicate thinking effectively through the effective use of mathematical language and terminology.
- To develop numeracy and literacy outlined in the 2011-2012 national plan

***Short-term goals …..(content goals specific to the lesson)* For students to review and apply**

- Coordinate geometry formulas to find length, slope, midpoint etc.
- Their knowledge of properties of parallelograms and special case parallelograms • Coordinate geometry formulas to find length, slope, midpoint etc.
- Their knowledge of circle theorems
- Their knowledge of congruent triangles and how it relates to parallelograms
- The use of Pythgoras' Theorem to prove that a triangle is right-angled.

4. Learning Outcomes:

As a result of studying this topic, students will be able to:

- Understand and recall the properties of a parallelograms and special cases • Effectively use the coordinate geometry formulae for distance, slope, midpoint.
- Recall what is required to prove that two triangles are congruent.
- Make connections between synthetic geometry and coordinate geometry
- Apply many different strategies to find slopes of line segments
- Recall that the distance formula is derived from Pythagoras' Theorem.

5. Background and Rationale

From the Junior Certificate Mathematics Syllabus (to be examined from 2016), the following are particularly relevant to the problem presented to students in this lesson.

On developing skills and understanding:
"*The objectives of Junior Certificate Mathematics are that learners develop mathematical proficiency, characterized as…*
…strategic competence - ability to formulate, represent, and solve mathematical problems in both familiar and unfamiliar contexts
….adaptive reasoning—capacity for logical thought, reflection, explanation, justification and communication…"

On making connections between different topics:
"In each strand, and at each syllabus level, emphasis should be placed on making connections between the strands and on appropriate contexts and applications of mathematics so that learners can appreciate its relevance to current and future life. The focus should be on the learner understanding the concepts involved, building from the concrete to the abstract and from the informal to the formal."

On problem-solving:
"As outlined in the syllabus objectives and learning outcomes, the learner's experiences in the study of mathematics should contribute to the development of problem-solving skills through the application of mathematical knowledge and skills."

Also:
"Problem-solving tasks activate creative mathematical thinking processes as opposed to imitative thinking processes activated by routine tasks. Reasoning mathematically about tasks empowers learners to make connections within mathematics and to develop deep conceptual understanding".

"*In the course of studying this strand [Strand 2 Geometry and Trigonometry], the learner will …*

- *solve geometrical problems and in some cases present logical proofs*
- *interpret information presented in graphical and pictorial form*
- *analyse and process information presented in unfamiliar contexts*
- *select appropriate formulae and techniques to solve problems."*

This study also addresses the following more general concerns regarding students in Maths class. Students (and teachers) have a tendency to compartmentalize their learning of Maths, and lose sight of the connections between the various approaches and types of Maths.

Students also tend to assume that there is one correct answer, and one correct way of finding that answer.

Students have difficulty using the language of mathematics and are often uncomfortable communicating a line of reasoning, either in writing or verbally.

Similar triangles are often confused with congruent triangles, and there is a misconception that proving the former implies the latter.

6. Research

- Junior Certificate Maths syllabus for 2016 onwards (NCCA, 2012)
- First, second and third year handbooks for Junior Cert Maths (Maths Development Team, revised editions 2016.
- "Deepening the understanding of the properties of a parallelogram". Lesson Study presented at Maths Counts 2015 by Sandra Gilmore, Thérése Ruane, Deirdre Newell and Lynn Anderson, Tiernan's Community School.

7. About the Unit and the Lesson

The SEC's Chief Examiners report on the Junior Certificate Maths Examination (2015) indicates that geometry and trigonometry questions were poorly answered. Students "struggled with topics from Strand 2 (Geometry and Trigonometry,…with trigonometry causing particular difficulty". The Chief examiner goes on to suggest that students' strategic competence was lacking when it came to geometry and trigonometry, with students finding it difficult to identify the approach (geometry, trigonometry, algebra etc). that was required for particular problems, especially unfamiliar or out-of-context problems.

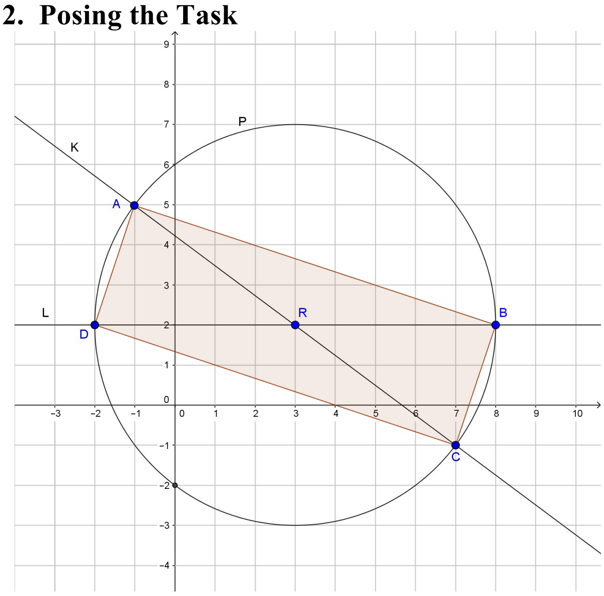
It is hoped that through their exposure to problems such as that presented here, students will improve their ability to analyse a problem from different perspective and identify which aspects of their mathematical knowledge may be used to solve the problem. Furthermore, by asking students to come up with a variety of possible approaches, they are encouraged to be as broad as possible in asking themselves "what do I know and what skills do I have, that I could use to help in coming up with a solution to this problem".

8. Flow of the Unit:

Lesson		# of lesson periods
1	• Review of Parallelograms and Congruent Triangles	1 x 45 min.
2	• Review of Circle Theorems	1 x 45 min.

3	• *Coordinate Geometry meets Synthetic Geometry*	1 x 60 min. **(research lesson)**
4	• Analysing further strategies on the research lesson problem.	1 x 45 min.
5	• Review of Formal Proofs	1 x 45 min.

9. Flow of the Lesson

<table>
<tr><th>Teaching Activity</th><th>Points of Consideration</th></tr>
<tr><td></td><td></td></tr>
<tr><td>1. Introduction
Review of the following prior knowledge:
1. Properties of Quadrilaterals and Parallelograms
2. Review of Coordinate Geometry formulae and strategies for finding slopes.
3. Congruent triangles
4. Circle Theorems and their corollaries
5. Transformations (translation, axial symmetry)</td><td>5 minutes
Much of the prior knowledge review will have been carried out in the preceding lessons.</td></tr>
<tr><td>2. Posing the Task

</td><td>5 minutes
Students are told that their task is to find as many ways as possible to show that ABCD is a rectangle.

The emphasis is on strategies, rather than calculations, and this must be made clear to them. They will also be told that there may be overlapping strategies, ones that might have some steps in common, but then diverge. Variety
is the what is being looked for. They are encouraged to use their imaginations and be creative.</td></tr>
<tr><td>Given: ABCD is a quadrilateral. Points A, B, C, D are on circle P which has centre R. Line K contains the points A, R and C. Line L contains the points D, R and B.

To Prove: ABCD is a rectangle.</td><td></td></tr>
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3. Anticipated Student Responses

1. Using properties of parallelograms:

To show that ABCD is a parallelogram:

- both pairs of opposite sides are equal in length (distance formula)
- both pairs of opposite sides are parallel (Slope; formula, rise/run, y=mx+c)
- one pair of sides is parallel and equal in length (as above)
- the diagonals bisect each other (midpoint/distance formula)
- the opposite pairs of angles are equal (perpendicular slopes $m_1.m_2 = -1$)

To show it is a rectangle:

- Diagonals are equal in length (radii/diameters, distance formula *or* Pythagoras'), therefore it is a rectangle.
- One of the angles is 90° as shown by one of the following methods: o slope formula using 2 points
    - rise/run
    - slope from equation of the line y = mx + c o Pythagoras corollary o Circle theorem
    - Trigonometry to find angles within both right angled triangles, two of which add to give 90°

Then, argue that since the opposite angle is also 90° (property of parallelograms,) and the other two must add up to 180 (sum of angles equals 360) and equal one another (opposite angles) etc.

Or, argue that since it is a parallelogram, then interior angles add to 180° etc.

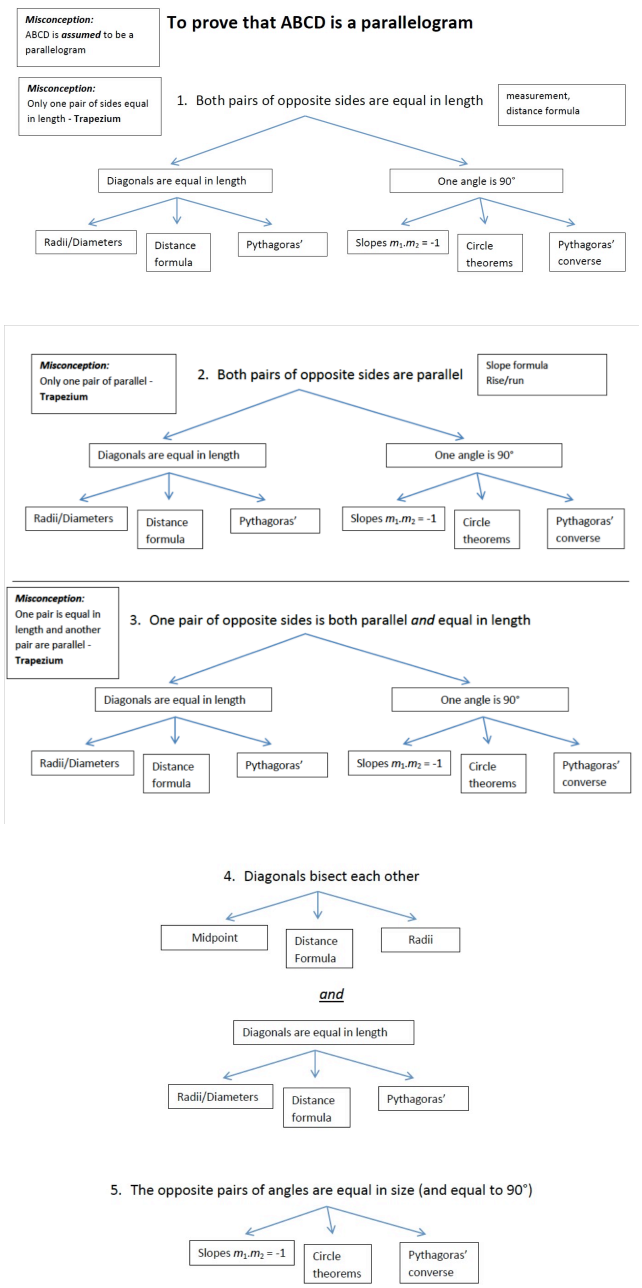
2. Congruent triangles:

Without arguing that ABCD is a parallelogram, congruency of the triangles formed can be used:
SAS
ASA
SSS

Length of sides found using distance formula, or argued following work to show that ABCD is a parallelogram.

25 minutes

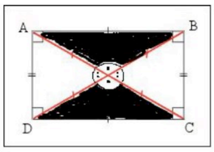
Many of the proofs have some steps in common. To facilitate observers in identifying substantially different approaches, the following flow diagram was designed.



3. **Three of the angles are 90°** (and therefore the 4th, since it is a quadrilateral).
Using strategies as in 1 above.

4. **Constructing perpendicular bisectors of perpendicular sides and showing that they are axes of symmetry.**
Mapping of points ABCD onto one another can be used to demonstrate symmetry.

6. Congruent Triangles



SAS
Radii.
Opposite angles.

and

One angle is 90°

Slopes $m_1.m_2 = -1$ | Circle theorems | Pythagoras' converse

7. Congruent triangles

Radii/Diameters | Distance formula | Measurement

SSS

and

One angle is 90°

Slopes $m_1.m_2 = -1$ | Circle theorems | Pythagoras' converse

Other proofs (without initial proof of a parallelogram)

1. Three of the angles are 90° (and therefore the 4th)

Slopes $m_1.m_2 = -1$ | Circle theorems | Pythagoras' converse

2. One pair of sides is parallel and one angle is 90°

Slope formula | Rise/run | Slopes $m_1.m_2 = -1$ | Circle theorems | Pythagoras' converse

and

Consecutive angles/complementary angles (transversal on parallel lines)

or

Further use of Circle theorems

3. Construct a perpendicular bisector of both perpendicular sides and show that both are axes of symmetry

and

one angle is 90°

Slopes $m_1.m_2 = -1$ | Circle theorems | Pythagoras' converse

Misconception:
A **rhombus** has two lines of symmetry too.

	Possible misconceptions Showing one pair of sides is parallel as proof of a parallelogram (a trapezium is possible). One pair of sides is parallel and the other pair of sides is equal in length (a trapezium also possible). Diagonals bisect one another (parallelogram) *or* equal in length (also true of a trapezium) not sufficient to show rectangle. One axis of symmetry is insufficient (trapezium, kite).
4. Comparing and Discussing Students who first prove that ABCD is a parallelogram will be asked to present first. Different approaches to finding slopes and distances will be identified during the previous session, and students will be selected so that a variety of approaches can be presented. Those who solely use a coord geom approach will be asked next. 3 or 4 students using a variety of strategies will present and their work will be grouped. If anyone has used circle theorems etc., then they will be presented. Those who use congruent triangles will be asked next.	**20 minutes**
5. Summing up	**5 minutes**

10. Evaluation

Observers will circulate and take notes on student progress, misconceptions, difficulties and trends. They will also be on the look-out for particular solutions that we have identified as those we would like to have students present in the Ceardaiocht session. Notes will be taken on a Lessson Observation Sheet, and times of observation notes will be recorded. Each teacher will have a seating plan in order to identify the students whose work they are observing and taking notes on.

Photographs of student work will be taken, and audio recordings of student conversations will also be made. The time that these are recorded will also be noted.

In general, teachers will not assist students or answer questions. This includes the teacher who will lead the lesson. Questions may be asked of students in order to clarify the work that they are involved with.

During the discussion part of the lesson, pre-selected students will be asked to come up to the board in a certain order to present their solutions. The solutions they present and the order in which they are asked to do so will have been decided upon before the lesson.

Observing teachers will also be looking for possible misconceptions that have been identified.

11. Board Plan

The following posters will be put up:
- Coordinate Geometry formulas
- Circle Theorems and their corollaries
- Types of quadrilaterals
- Properties of parallelograms
- Trigonometric ratios
- Pythagoras' Theorem and corollary
- Properties of Angles

Main headings will include:

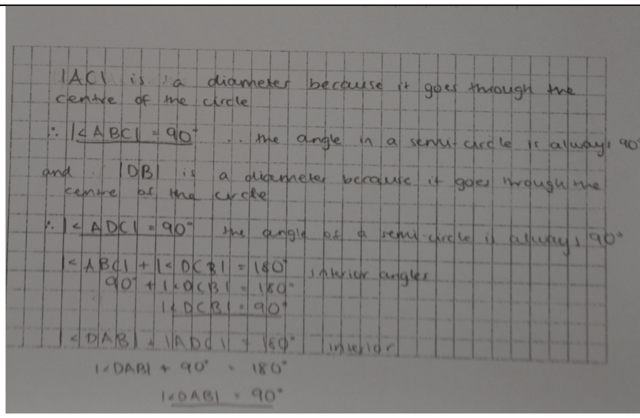
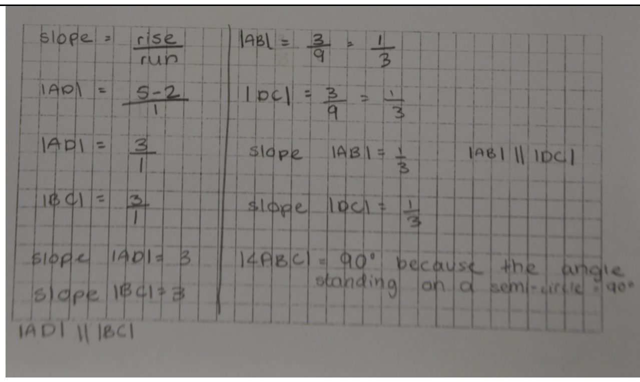
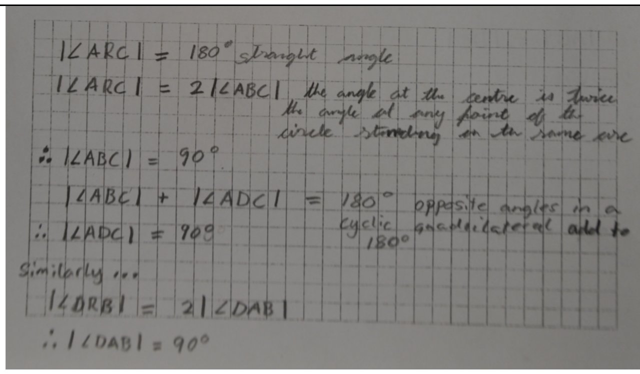
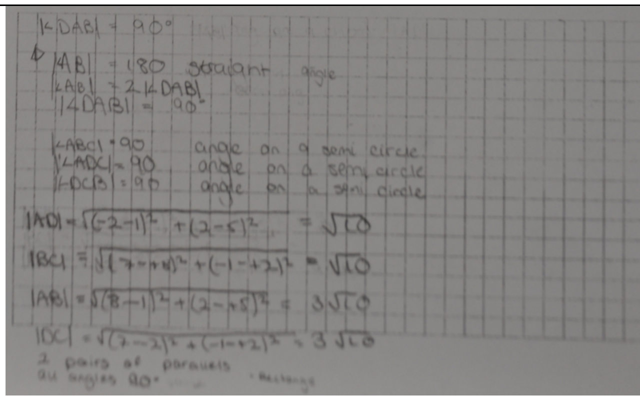
By proving ABCD is a parallelogram
- Both pairs of opposite sides are equal in length
- Both pairs of opposite sides are parallel
- Diagonals bisect each other
- One pair of opposite sides is both parallel *and* equal in length
- The opposite pairs of angles are equal in size (and equal to 90°)
- Congruent Triangles

Without proving that ABCD is a parallelogram

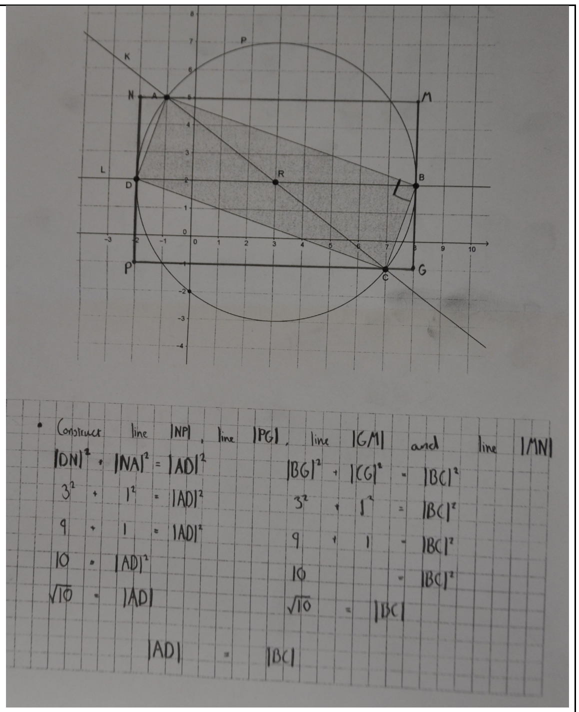
- Three of the angles are 90° (and therefore the 4^{th})
- One pair of sides is parallel and one angle is 90°
- Construct a perpendicular bisector of perpendicular sides and show that both are axes of symmetry

12. Post-lesson reflection

The main areas of which observers agreed was evident were;
- Students were engaged, enthusiastic about solving the problem.
- Students discussed and debated methods of solving the problem openly which was encouraging to see.
- The goals of the lesson were achieved.
- There was some excellent examples of students Maths literacy including the use of "similarly" and explanation of Circle theorems.
- Many different approaches were taken by all the students.

Sample student responses	
Student Response 1: Angle on a semi circle	 |AC| is a diameter because it goes through the centre of the circle ∴ |∠ABC| = 90° ∵ the angle in a semi-circle is always 90° and |DB| is a diameter because it goes through the centre of the circle ∴ |∠ADC| = 90° the angle of a semi-circle is always 90° |∠ABC| + |∠DCB| = 180° interior angles 90° + |∠DCB| = 180° |∠DCB| = 90° |∠DAB| + |ADC| = 180° interior |∠DAB| + 90° = 180° |∠DAB| = 90°
Student response 2: Perpendicular Slopes	 slope = rise/run |AD| = (5-2)/1 |AD| = 3/1 |BC| = 3/1 slope |AD| = 3 slope |BC| = 3 |AD| || |BC| |AB| = 3/9 = 1/3 |DC| = 3/9 = 1/3 slope |AB| = 1/3 |AB| || |DC| slope |DC| = 1/3 |∠ABC| = 90° because the angle standing on a semi-circle = 90°
Student response 3: Angle on a semicircle (one right angle) Cyclic quadrilateral (further right angles)	 |∠ARC| = 180° straight angle |∠ARC| = 2|∠ABC| the angle at the centre is twice the angle at any point of the circle standing on the same arc ∴ |∠ABC| = 90° |∠ABC| + |∠ADC| = 180° opposite angles in a cyclic quadrilateral add to 180° ∴ |∠ADC| = 90° Similarly ... |∠DRB| = 2|∠DAB| ∴ |∠DAB| = 90°
Student response 4: Length of opposite sides equal (parallelogram) and one right angle (circle theorem)	 |∠DAB| = 90° |AB| = 180 straight angle |AB| = 2|∠DAB| |∠DAB| = 90° |∠ABC| = 90 angle on a semi circle |∠ADC| = 90 angle on a semi circle |∠DCB| = 90 angle on a semi circle |AD| = √((-2-1)² + (2-5)²) = √10 |BC| = √((7-4)² + (-1-2)²) = √10 |AB| = √((8-1)² + (2-5)²) = 3√10 |DC| = √((7-2)² + (-1-2)²) = 3√10 2 pairs of parallels all angles 90° ∴ Rectangle

Student response 5: Using Pythagoras to calculate length of opposite sides (parallelogram proof)



Ways in which the lesson could be revised;

- The board work may need to be enlarged for all students to see the different solutions. A possible solution would be to use a visualizer or other method of projection.
- Some time (5 mins) could be taken from setting out prior knowledge and added to the board work.
- It was agreed that 4-5 solutions should be presented on the board by students – ideally one from each of the main categories of proofs identified.
- Ways of summarising the lesson were discussed as time pressures caused the lesson to finish during the board work.