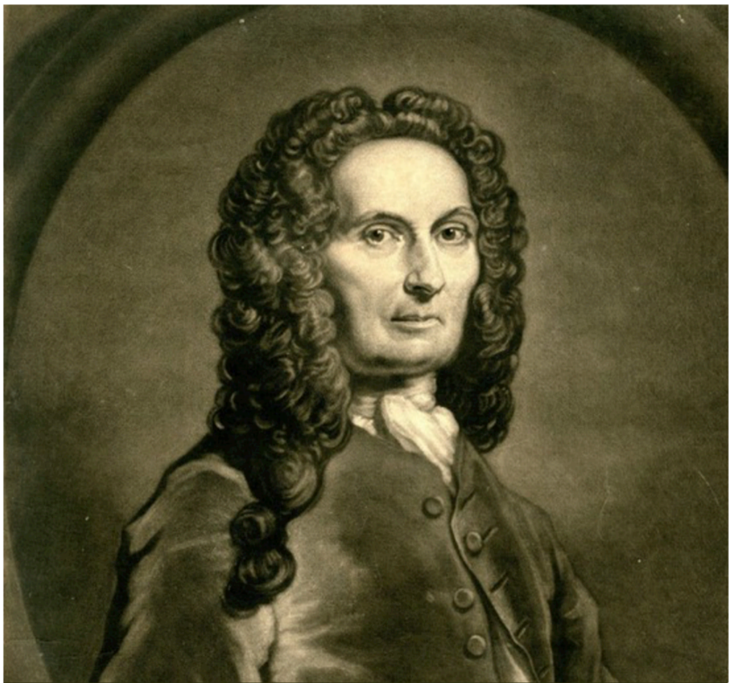
“Decoding De Moivre”

Year Group: Transition Year Higher Level
Topic: Complex Numbers – De Moivre’s Theorem



Abraham de Moivre
(1667-1754)

Lesson Study and Research Group:

David Crowdle and Paudge Brennan, teachers in Loreto SS, Wexford, developed this lesson proposal. Paudge taught the lesson on 10th February 2017 with 17 students from his Transition Year class.

Brief description of the lesson:

The research lesson is the second structured problem solving classes leading the students to appreciate and develop an understanding of polar multiplication and raising a number to a power (De Moivre’s Theorem) in terms of the moduli and arguments of the numbers. Students will engage with a problem working out the value of a given complex number raised to the power of 5. Together they will present approaches leading to an understanding of de Moivre’s Theorem and the need for it.

Aims of the Lesson:

Short term aim: Students will:

- appreciate that there are different correct approaches to expanding a complex number,
- see the advantages to using the "Modulus-argument" form of a complex number when it comes to multiplying,
- develop a conceptual understand the principle of de Moivre's theorem "to expand a number raised to a power you raise the modulus to that power and multiply the argument by the power."[1]

Long term aims: We would like our students:

- to gain confidence in dealing with abstract concepts,
- to become independent learners through structured problem solving,
- to connect and review the concepts that we have studied already,
- to develop their literacy and numeracy skills through discussing ideas. [1]
- to develop key skills such as "Managing Information & Thinking: Thinking creatively and critically",[2]

Learning Outcomes:

As a result of studying this topic students will be able to:

- Understand in different ways the meaning of multiplication of whole numbers and use this to make sense of complex number multiplication and expansion.
- Understand and be able to express a complex numbers both in rectangular form and in terms of its modulus and argument.
- Recognise a number on an Argand diagram in terms of its modulus and argument
- Develop the insight that when numbers are multiplied their moduli are multiplied and their arguments are added together.
- Be able to multiply numbers given their arguments and moduli.
- Use this to discover that when a number is raised to a power its modulus is raised to that power and its argument is multiplied by that power and experience that this approach is much more efficient that using repeated multiplication in rectangular form.

Background and Rationale

This topic was chosen because students find dealing with the polar form of complex numbers and De Moivre's Theorem a challenging part of the Higher Level Course. It is hoped that this approach will help students appreciate the efficiency of using the polar form for multiplication and applying powers to numbers.

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[1] Department of Education & Skills (2011). *Literacy and Numeracy for Learning and Life: the National Strategy to Improve Literacy and Numeracy among Children and Young People 2011-2020.*

[2] NCCA (2015). *Junior Cycle Key Skills: Managing Information and Thinking*, http://www.juniorcycle.ie/NCCA_JuniorCycle/media/NCCA/Documents/Key/Managing-information-and-thinking_April-2015.docx, Accessed 12 February 2017

Research

- NCCA (2014). *Leaving Certificate Mathematics Syllabus – Foundation, Ordinary and Higher Level for Examination from 2015*
- (Project Maths Development Team (2015). *Teacher Handbook, Senior Cycle Ordinary Level*. http://www.projectmaths.ie/documents/handbooks/LCOLHandbook2015.pdf
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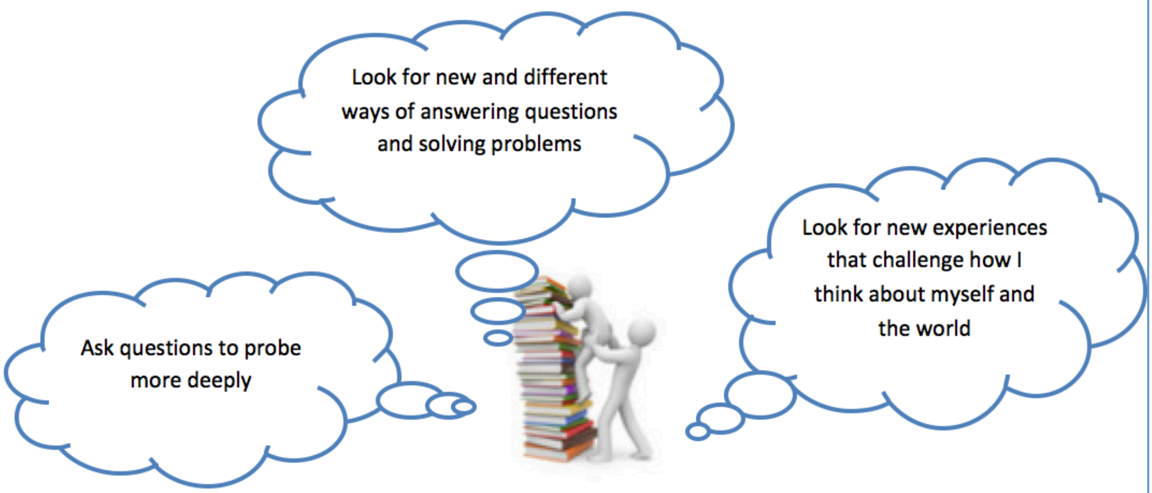
- Toshiakira FUJII (2013). *The Critical Role of Task Design in Lesson Stud*y. ICMI Study 22: Task Design in Mathematics Education
- Maths Development Team, *Reflections on Practice from Maths Counts 201*6: http://www.projectmaths.ie/for-teachers/conferences/maths-counts-2016/
- Department of Education & Skills (2011). *Literacy and Numeracy for Learning and Life: the National Strategy to Improve Literacy and Numeracy among Children and Young People 2011-2020*
- NCCA, *Junior Cycle Key Skills: Managing Information and Thinking*, http://www.juniorcycle.ie/NCCA_JuniorCycle/media/NCCA/Documents/Key/Managing-information-and-thinking_April-2015.docx.



Supportive Classrooms

Let's begin by thinking about classroom **cultures,** because establishing the right climate is a crucial factor in encouraging curiosity. Students need a climate where they feel comfortable about being curious, asking questions and making mistakes.

About the Unit and the Lesson

According to the syllabus with regard to Complex Numbers students are required to:

ORDINARY LEVEL

- Investigate the operations of addition, multiplication, subtraction and division with complex numbers C in rectangular form a + ib.
- Illustrate complex numbers on an Argand diagram.
- Interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate.

HIGHER LEVEL

- Geometrically construct root 2 and root 3.
- Calculate conjugates of sums and products of complex numbers.
- Use the Conjugate Root Theorem.
- Work with complex numbers in rectangular and polar form to solve quadratic and other equations.
- Use De Moivre's Theorem.
- Prove De Moivre's Theorem by induction for n an element of N.
- Use applications such as nth roots of unity, n an element of N, and identities such as $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

With this Transition Year group only the Ordinary Level material and an informal approach to the polar form of a complex number and related multiplication and deMoivre's Theorem will be covered.

Flow of the Unit:

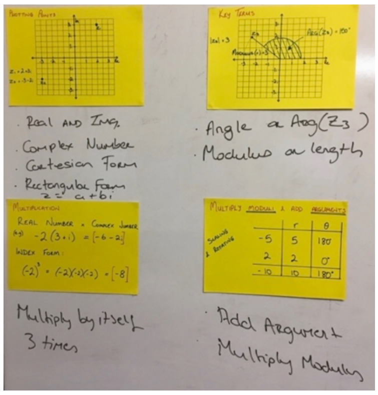
No.	Lesson	Time
1	Introduction to Complex Numbers and the Argand diagram	40 min.
2, 3	Addition and subtraction of complex numbers	40 min.
4, 5	Complex Multiplication	40 min.
6	Modulus of a complex number	40 min.
7, 8	Complex Division	40 min.
9	Expressing numbers in terms of argument and modulus. Consider integer multiplication in terms of "scaling and rotation."	40 min.
10	Multiplication (Structured problem solving lesson): consider scaling in terms of the moduli and the rotation in terms of the arguments. (Appendix II)	40 min.
11	Reserch Lesson: "Decoding DeMoivre"	40 min.

Flow of the Lesson

1. Introduction (5 minutes)

Prior Knowledge:
Students are questioned on their understanding of previous classes concerning:

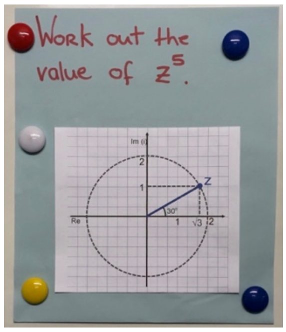
a) Argand diagram
b) Rectangular form of a complex number
c) Describing the argument and modulus of a complex number
d) Indices
e) Multiplication by expansion
f) Multiplication in terms of arguments and moduli



2. Posing the Task (2 minutes)

The task is presented on the board. They are instructed to find the value of z^5 in “as many ways as you can” in “15 minutes”. Students are questioned on their understanding of the task.

Each student is given 3 copies of the handout page with the question and they are asked to “explain their thinking” (Appendix I).

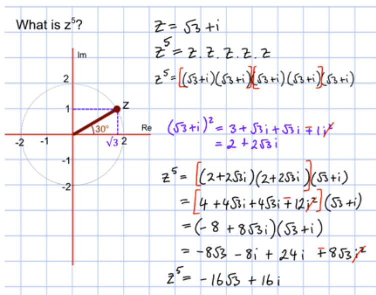


3. Individual problem solving (15 minutes)

Anticipated Student Response 1

It is expected that most students will recognise that the rectangular form of z = √3 + i and then they might try to evaluate $(\sqrt{3} + i)^5$ by expanding it in a traditional way – repeated multiplication.

Solution 1 - Using rectangular form multiplication

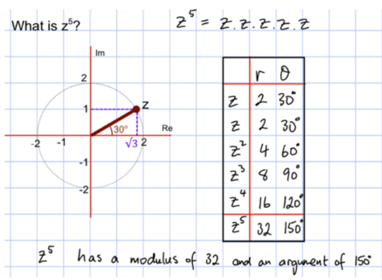


Anticipated Student Response 2

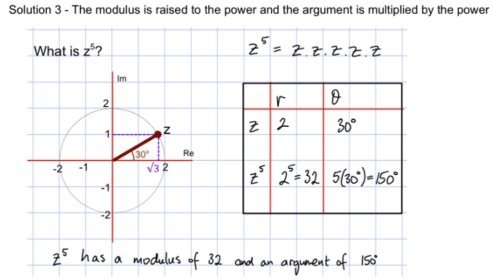
It is hoped that students might try to describe the modulus and argument of z. |z| = 2 and Arg (z) = 30°

They might then describe the expansion of z^5 in terms of repeated multiplication:
$|z^5| = (2)(2)(2)(2)(2) = 32$
$\text{Arg}(z^5) = 30^\circ+30^\circ+30^\circ+30^\circ+30^\circ = 150^\circ$

Solution 2 - Multiply the moduli and add the arguments



<table>
<tr><td></td><td></td></tr>
<tr><td>Anticipated Student Response 3

Students might build on "Anticipated Response 2" to express the expansion of z^5 in a more efficient way:
$|z^5| = (2)^5 = 32$
$\text{Arg}\,(z^5) = 5(30^\circ)$</td><td>
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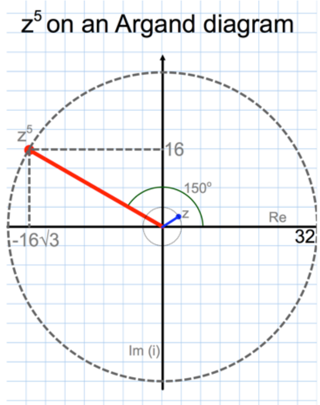
<tr><td>4 Presenting Student Responses and Ceardaíocht

The students will present their solutions and explain their thinking at the board. Any students may be asked to re-explain what has been presented. Misconceptions will be addressed.

The solutions will be presented in the in order of the Anticipated Responses (outlined above).

Students will see on an Argand diagram how the Rectangular form and the Argument/Modulus solutions for the expansion describe the same number.

It is hope that students might articulate: "A very efficient way of raising a number to a power is to raise the modulus by the power and multiply the argument by the power."</td><td>
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<tr><td>5. Summing up

It will be explained that the ideas discussed in todays class relate to de Moivre's Theorem

Students will be asked to write down whet they learned in today's class.</td><td>De Moivre's Theorem

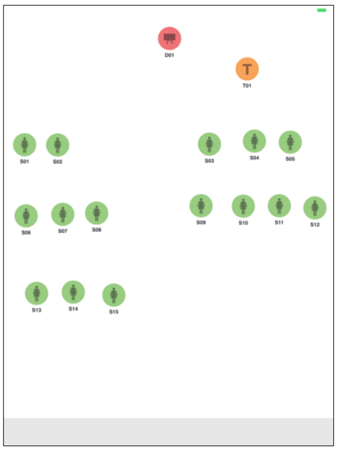
</td></tr>
</table>

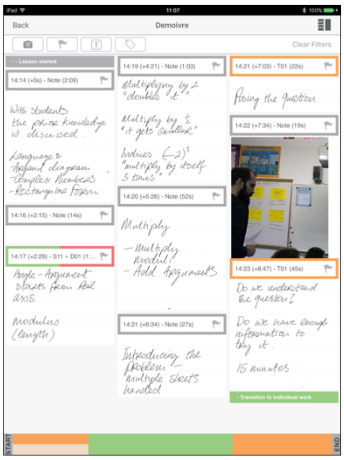
Plan for Observing the Students

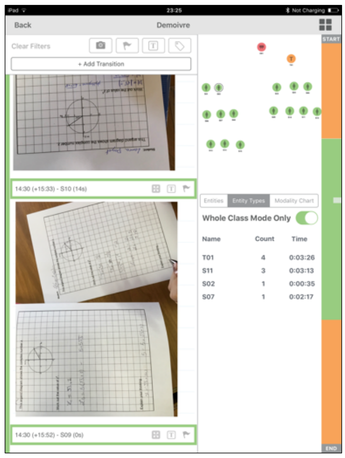
The teacher and two observers will be present for the research lesson. A seating chart will be used to identify students and their responses. The teacher will use a clip-board and record sheet that identifies which students attempted and completed each of the three anticipated responses. A camera will be used to record the board work. During the research lesson the teachers are prepared to consider the following Post-Lesson Discussion questions: What happened that you expected? What happened that you didn't expect? To what degree were the goals of the lesson achieved?

Introduction, posing the task	Can students recall prior knowledge relation to complex numbers? Was the task clear? Questions asked by students
Individual work	Can students correctly describe z in rectangular form? Can students correctly expand z in rectangular form? Can students correctly expand z^5 in terms of its argument and modulus? Can student correctly expand z^5 in terms of its argument and modulus? What problems do thestudents encounter? Are prompts required? What strategies do they employ when drawing congruent triangles? What kind of questions do students ask? Do they persist with the task?
Presentation and Discussion	Are students attentive to what is happening on the board? Are clarifications needed to presenters' board work? Did the discussion and boardwork promote student learning?

During each part of the class evidence will be recorded on paper and using the LessonNote App.



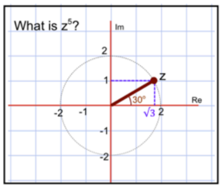




The answer sheets used by the student will be collected along with the students' reflection their own learning at the end of the lesson.

The Board Plan

Planned Boardwork



Solution 1 - Multiply (expand the brackets)

$z = \sqrt{3} + i$

$z^5 = z.z.z.z.z$

$z^5 = [(\sqrt{3}+i)(\sqrt{3}+i)][(\sqrt{3}+i)(\sqrt{3}+i)](\sqrt{3}+i)$

$(\sqrt{3}+i)^2 = 3 + \sqrt{3}i + \sqrt{3}i \mp 1 \cancel{i^2}$

$= 2 + 2\sqrt{3}i$

$z^5 = [(2+2\sqrt{3}i)(2+2\sqrt{3}i)](\sqrt{3}+i)$

$= [4 + 4\sqrt{3}i + 4\sqrt{3}i \mp 12\cancel{i^2}](\sqrt{3}+i)$

$= (-8 + 8\sqrt{3}i)(\sqrt{3}+i)$

$= -8\sqrt{3} - 8i + 24i \mp 8\sqrt{3}\cancel{i^2}$

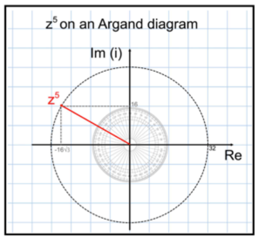
$z^5 = -16\sqrt{3} + 16i$

Solution 2 - Multiply the moduli and add the arguments (5 times)

	r	θ
z	2	30°
z	2	30°
z^2	4	60°
z^3	8	90°
z^4	16	120°
z^5	32	150°

Solution 3 - The modulus is raised to the power of 5 and the argument is multiplied by 5

	r	θ
z	2	30°
z^5	$2^5 = 32$	$5(30°) = 150°$



What is have you learned?

An efficient way of raising a number to a power is to raise the modulus by the power and multiply the argument by the power.

DeMoivre's Theorem

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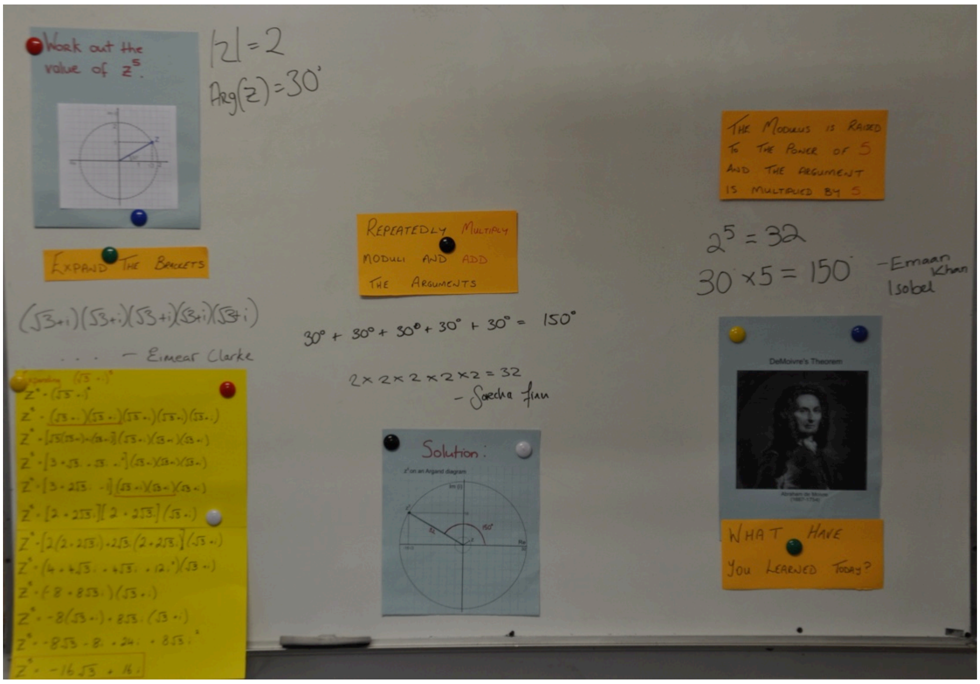
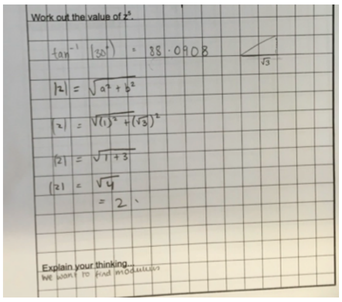
Actual Boardwork

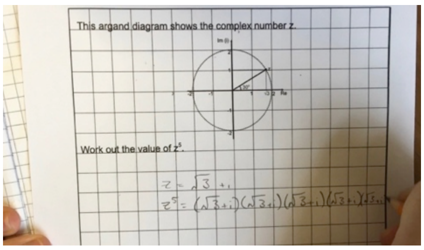


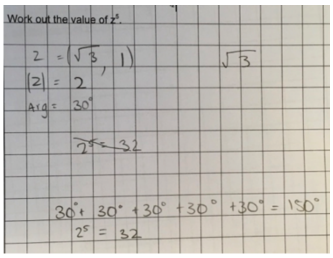
Post Lesson Reflection

- 15 students correctly interpreted and wrote down the rectangular form of z.
- 14 students tried to expand z^5 using repeated multiplication, but no student completed this correctly. Discussion around problems relating to this expansion supported the learning in the class.
- A number of students in their efforts to complete this method correctly did not have time to consider other approaches.
- 2 students had the misconception that $(\sqrt{3} + i)^5 = ((\sqrt{3})^5 + (i)^5$
- A number of students spent time working out the modulus and the argument even though this information could be gleaned from the diagram. When a student pointed this out, there was a collective "Oooh!" from the class.



- 4 students used the modus and the argument to correctly expand z^5.
- The students were able to derive and explain all three anticipated responses. Although questioning was used to lead students to write that $30^\circ+30^\circ+30^\circ+30^\circ+30^\circ$ could more efficiently be described as $5(30^\circ)$.



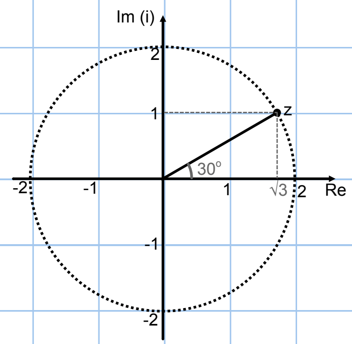


- During Ceardaíocht a student asked: "What if the power isn't 5?" Other students were able to generalise how to apply the principle of de Moivre's Theorem.
- The learning goals of the class seemed to be achieved. 10 student reflection responses stated in one-way or another that: "to multiply complex numbers you multiply the moduli and add the arguments" or to "raise a complex number by a power you raise the modulus by the power and multiply the argument by the power".
- Many expressed an appreciation for this method: "I learned that a page of maths could just as easily be solved inn two lines - $|z^5| = (2)^5 = 32$ and Arg $(z^5) = 5 \times 30^o$" or that "it is easier to multiply complex numbers now".

Appendix I –Student Handout with the Problem

Student:

This argand diagram shows the complex number z.

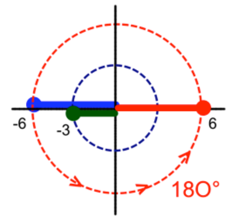


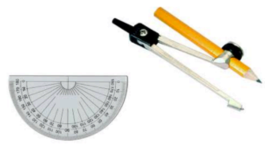
Work out the value of z^5.

Explain your thinking...

Appendix II – Student Tasks Prior to this Lesson

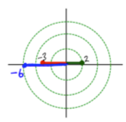
Students understand integer multiplication
- By rule: eg (-2)(-3) = 6
- By scaling and rotating: eg (-2)(-3) means that (-3) is scaled by a factor of 2 rotated clockwise by 180°





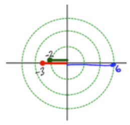
Task 1 - Understanding multiplication in terms of moduli and arguments

Express each of these numbers in terms of their modulus and argument:
(i) 2 (ii) -3 (iii) (2)(-3)



	modulus	argument
2	2	0°
-3	3	180°
(2)(-3)	6	180°

Express each of these numbers in terms of their modulus and argument:
(i) -2 (ii) -3 (iii) (-2)(-3)



	modulus	argument
-2	2	180°
-3	3	180°
(-2)(-3)	6	360° (or 0°)

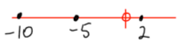
What is happens when you multiply two numbers in terms of their moduli and arguments?

The moduli are multiplied and the arguments are added.

Task 2 - Understanding multiplication in terms of moduli and arguments

Can you come up with your own example to show that this is true (or otherwise).

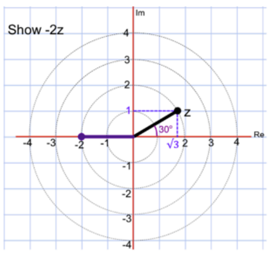
$(-5)(2) = -10$



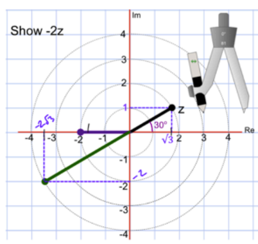
	r	θ
-5	5	180°
2	2	0°
-10	10	180°

It's true in this example

Task 3 - Complex Multiplication using different methods



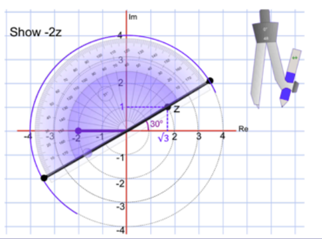
Solution 1 - Using rectangular form multiplication



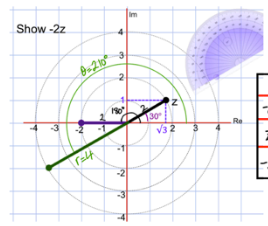
$z = \sqrt{3} + i$
$-2z = -2(\sqrt{3}+i)$
$= -2\sqrt{3} - 2i$

Students may:
- use a decimal approximation for $2\sqrt{3}$
- or construct it.

Solution 2 - Construct solution by scaling and rotating



Solution 3 - Multiply the moduli and add the arguments



	r	θ
-2	2	180°
z	2	30°
-2z	4	210°