Teaching & Learning Plans

Geometric Series

Leaving Certificate Syllabus
Higher level

Project Maths
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Development Team

The Teaching & Learning Plans are structured as follows:

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Aims outline what the lesson, or series of lessons, hopes to achieve.

Prior Knowledge points to relevant knowledge students may already have and also to knowledge which may be necessary in order to support them in accessing this new topic.

Learning Outcomes outline what a student will be able to do, know and understand having completed the topic.

Relationship to Syllabus refers to the relevant section of either the Junior and/or Leaving Certificate Syllabus.

Resources Required lists the resources which will be needed in the teaching and learning of a particular topic.

Introducing the topic (in some plans only) outlines an approach to introducing the topic.

Lesson Interaction is set out under four sub-headings:

i. Student Learning Tasks – Teacher Input: This section focuses on possible lines of inquiry and gives details of the key student tasks and teacher questions which move the lesson forward.

ii. Student Activities – Possible Responses: Gives details of possible student reactions and responses and possible misconceptions students may have.

iii. Teacher's Support and Actions: Gives details of teacher actions designed to support and scaffold student learning.

iv. Assessing the Learning: Suggests questions a teacher might ask to evaluate whether the goals/learning outcomes are being/have been achieved. This evaluation will inform and direct the teaching and learning activities of the next class(es).

Student Activities linked to the lesson(s) are provided at the end of each plan.

Teaching & Learning Plans: Geometric Series

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Aims

The aim of this series of lessons is to enable students to:

- understand the concept of a geometric series
- use and manipulate the appropriate formula
- apply their knowledge of geometric series to everyday applications
- deal with combinations of geometric sequences and series and derive information from them
- find the sum to infinity of a geometric series, where $-1 < r < 1$
- use the concept of the sum to infinity and the corresponding formula to solve problems

Prior Knowledge

It is envisaged that in advance of tackling this Teaching and Learning Plan, the students will understand and be able to carry out operations in relation to:

- the concept of pattern
- basic number systems
- graphs of functions in the co-ordinate plane
- simultaneous equations of two unknowns
- indices
- proof by induction
- logarithms

Students should also have completed the following Teaching and Learning Plans: Arithmetic Sequences, Arithmetic Series and Geometric Sequences

Learning Outcomes

Having completed this Teaching and Learning Plan the students will be able to:

- recognise geometric series and their everyday applications
- recognise series that are not geometric
- apply their knowledge of geometric series in a variety of contexts
- apply and manipulate the relevant formulas in both theoretical and practical situations
- find the sum to infinity of a geometric series, where $-1 < r < 1$
- apply their knowledge of infinite geometric series in a variety of contexts

Catering for Learner Diversity

In class, the needs of all students, whatever their level of ability level, are equally important. In daily classroom teaching, teachers can cater for different abilities by providing students with different activities and assignments graded according to levels of difficulty so that students can work on exercises that match their progress in learning. Less able students, may engage with the activities in a relatively straightforward way while the more able students should engage in more open–ended and challenging activities

In interacting with the whole class, teachers can make adjustments to meet the needs of all of the students. **For example, some students may engage with some of the more challenging questions for example question number 12 in Section A: Student Activity 1.**

Apart from whole-class teaching, teachers can utilise pair and group work to encourage peer interaction and to facilitate discussion. The use of different grouping arrangements in these lessons should help ensure that the needs of all students are met and that students are encouraged to articulate their mathematics openly and to share their learning.

Relationship to Leaving Certificate Syllabus

Sub-Topic	**Learning outcomes**		
Students learn about	**Students working at FL should be able to**	**In addition students working at OL should be able to**	**In addition students working at HL should be able to**
3.1			– investigate geometric sequences and series
3.1			– solve problems involving finite and infinite geometric series including applications such as recurring decimals – derive the formula for the sum to infinity of geometric series by considering the limit of a sequence of partial sums

Resources Required

- Whiteboard and markers
- The Student Activities contained in this T&L
- Access to a computer room to conduct the on line quiz on the last day of this series of lessons. Alternatively paper copies of the quiz can be printed out and used
- The free *GeoGebra* software package available at www.geogebra.org or another suitable functions graphing application if suitable hardware is available in the classroom.

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Teacher Reflections

Lesson Interaction			
Student Learning Tasks: Teacher Input	**Student Activities: Possible and Expected Responses**	**Teacher's Supports and Actions**	**Checking Understanding**
An Introduction to Geometric Series			
» Give me an example of a geometric sequence and tell me why you say it is geometric.	• {1, 3, 9, 27, 81,…} You are given a first term 1 and you multiply each term by 3 to get the next term. • {2, 4, 8, 16, 32,…} You are given a first term 2 and you multiply each term by 2 to get the next term.	» Write each example of a geometric sequence given by the students on the board and discuss why it is or is not a geometric sequence. » If students do not give an example of a sequence that is not geometric write {1, 2, 4, 6, 8,…} on the board and discuss why it is not geometric.	» Can students recognise why a particular sequence is or is not geometric? » Do the students recognise that for a geometric sequence $\frac{T_{n+1}}{T_n} = r$
» Write out the first eight terms of the geometric sequence {1, 2, 4,…} and give the corresponding series. » What is the sum to eight terms of this series?	• {1, 2, 4, 8, 16, 32, 64, 128} 1+2+4+8+16+32+64+128 • 255	» Write the sequence and series on the board and clearly label which is the sequence and which is the series. » The teacher points out that the sum to eight terms is an example of a partial sum of the series.	» Do students understand the difference between a sequence and a series? » Do they understand what is meant by the partial sum of a series?

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Finding the sum of the first 8 terms was easy but there is a formula that helps us especially when the sequence has a large number of terms. » The formula is $S_n = \frac{a(1-r^n)}{1-r}$ and it tells us the sum of the first n terms of the series, which is partial sum of a geometric series.		» Write the formula $S_n = \frac{a(1-r^n)}{1-r}$ on the board and explain what each of the terms mean. a = the first term of the series r = the common ratio n = the number of terms » Write S_8 on the board.	» Do students recognise what each letter in this formula represents? » Do students understand the difference in the formula $T_n = ar^{n-1}$ and the formula $S_n = \frac{a(1-r^n)}{1-r}$?
» This formula appears on page 22 of the formula and tables booklet. » How does this formula and $T_n = ar^{n-1}$ differ?	• T_n gives the n^{th} term of a geometric sequence and S_n gives the sum of the first n terms of a geometric series.	» Show students how to locate the formula on page 22 of the formula and tables booklet.	
» What happens to the terms of a geometric sequence when $\|r\|$ is greater than 1? » What happen to the terms of a geometric sequence when $\|r\|$ is less than 1? **Note:** sometimes if $\|r\| > 1$ the partial sum of a geometric formula can be written as $S_n = \frac{a(r^n-1)}{r-1}$. This is to make calculations easier, but you will get the same answer if you use $S_n = \frac{a(1-r^n)}{1-r}$ in all cases.	• Successive terms have a larger numerical value and their sign depends on the sign of r. • Successive terms have a smaller numerical value and their sign depends on the sign of r.	» Write both formulas on the board.	» Do students understand that both formulas give the same answer irrespective of the value of r?

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Do the first 3 questions in **Section A: Student Activity 1** using the formula for the sum to n terms		» Distribute **Section A: Student Activity 1.**	» Can the students apply the formula to the problems in the activity sheet?
» Now we are going to prove the formula $S_n = \frac{a(1-r^n)}{1-r}$ is true for all geometric series. **Note** the proof of this formula by induction is on the syllabus. » How does a proof by induction work?	• You prove the statement to be true for the first value of n, typically n is equal to one. • Then you assume it is true for say $n = k$. • Now assuming it is true $n = k$ you prove it is true for $n = k + 1$. • But you have proved it is true for your initial value hence it is true for all integer values greater than the initial value.	» Write each stage on the board in words.	» Do students understand what it means to prove something? » Do students understand the principle of proof by induction?
» How do we show this formula is true for $n = 1$?	• $S_1 = \frac{a(1-r^1)}{1-r} = \frac{a(1-r)}{1-r} = a$ Hence it is true for $n = 1$.	» Outline of proof is found in **Appendix A** of this teaching and learning plan. » Proof also available in PowerPoint format at http://www.projectmaths.ie/students/cd-strand3LC/strand3-number.asp under Induction and Proof of sum of Geometric Series.	» Do students understand the proof? » Do students understand that they have proved that this formula is true for all geometric series?

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Assuming it is true for $n = k$ what is S_k? » How do S_{k+1} differ from S_k? » Replacing S_k and T_{k+1} in this formula what do we get? » How can we now simplify this equation?	$S_k = \frac{a(1-r^k)}{1-r}$ $S_{k+1} = S_k + T_{k+1}$ $S_{k+1} = \frac{a(1-r^k)}{1-r} + a(r)^{(k+1)-1}$ $S_{k+1} = \frac{a(1-r^k)+ar^k(1-r)}{1-r}$ $S_{k+1} = \frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$ $S_{k+1} = \frac{a-ar^{k+1}}{1-r} = \frac{a(1-r^{k+1})}{1-r}$	» Allow students to give their suggested answers to each question on the opposite column 1.	
» What does this tell us about the formula and S_{k+1}? » So we proved the formula was true for $n = 1$ and $n = k + 1$ provided it is true for $n = k$, what does this tell us?	• It is true for S_{k+1} provided it is true for $n = k$. • It is true for $n = 1 + 1$, $n = 2 + 1$ and so on for all $n \in N$.		

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Continue with the questions on **Section A: Student Activity 1**.		» Circulate the room and offer assistance where needed. » If the majority of the students are unable to answer a particular question for themselves do that question on the board and through discussion enable students to gain an understanding of the question.	» Can students apply the formulas in the different contexts? » Do students understand the difference between T_n and S_n for a geometric sequence and series?
Wrap up and Homework. » What information does the formula $S_n = \frac{a(1-r^n)}{1-r}$ give us? » Does this formula work for all geometric series? » Provided we are given no further information about the equivalent sequence or the series itself what do we need to know in order to be able to apply the formula? » Questions not already attempted in **Section A: Student Activity 1** can be completed for homework.	• It tells us the sum of the first n terms of a geometric series, which is also known as a partial sum of the series. • Yes • At least three of the following a, r, n or S_n?	» Ensure that students understand that if they are given S_n and two of the following a, r or n then the S_n formula can be used to get the unknown. » It is also worth reinforcing that for all series $T_n = S_n - S_{n-1}$	» Are students able to use the formula? » Do students understand the purpose of the formula?

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
To write a series using sigma notation			
» How many terms are in the series $2 + 4 + 8 + \ldots 128$?	• 7	» Write the sequence on the board and then write all the terms in the form 2^x.	
» How would you get the 7th term of this series?	• 2^7		
» An alternative way of writing the partial sum of this series is $\sum_{n=1}^{7} 2^n$.		» The general term of the series is $T_n = 2^n$. » Write $\sum_{n=1}^{7} 2^n$ on the board.	
» The $\sum$ notation is known as sigma and means the sum of. The initial value of the unknown (n in this case) is placed underneath and the highest value of the unknown is placed on top of the sigma and the general term of the series is after the symbol.			» Do students understand that the series $1 + 3 + 9 + 27 + \ldots$ Can be written as $3^0 + 3^1 + 3^2 + \ldots$ and that the sum to eight terms (the partial sum) can be written as $\sum_{n=0}^{8} 3^n$?
» Write an expression for the partial sum ,to eight terms, of the series $1 + 3 + 9 + 27 + \ldots$ using $\sum$ notation.	• $\sum_{n=0}^{8} 3^n$	» Write the solution on the board.	
» Write the partial sum, to 50 terms, of the series x + x2 + x3 + … in $\sum$ notation	• $\sum_{n=1}^{50} x^n$	» Write the solution on the board.	» Can students use the sigma notation correctly?

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Evaluate $\sum_{n=1}^{6}(-1)^n 2^n$	• -2 + 4 - 8 + 16 - 32 + 64 = 42	» Write -2 + 4 - 8 + 16 - 32 + 64 = 42 on the board.	» Can students convert from sigma notation to the partial sum? » Do students understand that the series is an unevaluated sum and that $\sum_{n=1}^{6}(-1)^n 2^n$ is a number equal to the sum of its first 6 terms?
» Do exercises 1 to 6 in **Section B. Student Activity 2**.		» Distribute **Section B. Student Activity 2**. » Circulate the room and check for misunderstandings.	» Are students comfortable using the sigma notation?

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
To explore the fact that the partial sums of some series converge to a limit			
» What series does this represent $\sum_{n=1}^{\infty} 2^n$	• 2 + 4 + 8 + 16 + 32 + …	» It is a number representing the sum to infinity of the series.	» Do students understand the sigma notation?
» As n get bigger what happens to the corresponding term?	• It gets bigger.		» Can students differentiate between the concept of a term, a series and a partial sum?
» As n gets bigger what happens to the partial sums?	• They get bigger.		
» What series does this represent $\sum_{n=1}^{\infty} (\frac{1}{2})^n$	• 1/2 + 1/4 + 1/8 + 1/16 + …	» It represents the sum to infinity of the series. We are now going to see if we can find its value.	
» As n gets bigger what happens to the corresponding term?	• It gets smaller and smaller and eventually it approaches zero.		
» Discuss in pairs what happens to the partial sums of this series as n gets bigger.	• S_1 = 1/2 • S_2 = 1/2 + 1/4 = 3/4 • S_3 = 1/2 + 1/4 + 1/8 = 7/8 • S_4 = 1/2 + 1/4 + 1/8 + 1/16 = 15/16 • S_5 = 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 31/32 • $S_{20} = \frac{\frac{1}{2}(1-\frac{1}{2}^{20})}{1-\frac{1}{2}} = 0.999999046$	» Circulate the room as discussion is taking place and offer assistance when required. » Write the student responses on the board.	» Can students calculate the partial sums? » Do students recognise that the limit of the partial sums in this case is one?
» What conclusion did you arrive at regarding the partial sums as n gets bigger?	• They approach 1.		

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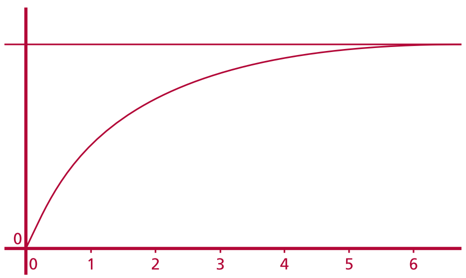
Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» We are now going to use *GeoGebra* to draw the function $f(x)\frac{\frac{1}{2}(1-(\frac{1}{2})^x)}{1-\frac{1}{2}}$ **Note:** x stands for the number of terms and $f(x)$ is the graph of the partial sums. » The graph of the function representing the partial sums would appear as follows: 		» Use the free software package GeoGebra available at www.geogebra.org or another suitable functions graphing software application if available. **Note** x stands for the number of terms and $f(x)$ is the graph that represents the partial sums of the series depending on the value of n.)	» Do students understand that the ratio of consecutive terms of the series must be between -1 and 1 in order for the partial sum of the series to converge to a particular number?
» What do we notice about $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n$? » We say this series converges to 1 because the partial sum of S_n approaches a number (1 in this case) and gets closer and closer to that number as n gets bigger. » The partial sum in this case will never be exactly 1, but as n gets bigger, the difference between the partial sum and 1 will be so small as to be insignificant.	• It approaches 0. • As n gets bigger the terms get smaller.	» If using the software adjust the ratio of the x axis: y axis to enable students to see the approach to the limit more accurately.	» Do students understand the difference between a series that converges and one that does not?
» If the partial sum of a particular series converges to a particular number does this mean consecutive terms are increasing or decreasing? Why? » When the partial sum of a series has a limit we say that the **series converges** to that limit or is **convergent**. » When the partial sum of a series has no limit we say that the series is divergent.	• Decreasing. As n gets bigger the consecutive terms get so small to make no significance difference to the partial sum.		

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Discuss in pairs what happens to the partial sums of this series as n gets bigger $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots$ » As n gets bigger do the partial sums converge to any particular number? » Depending on the decimal point setting on your calculator the partial sum of this series might show 3 but it will only have been rounded up to 3.	• $S_1=2$ $S_2=2\frac{2}{3}$ $S_3=2\frac{8}{9}$ $S_4=2\frac{26}{27}$ $S_5=2\frac{80}{81}$ $S_{20}=\dfrac{2(1-(\frac{1}{3})^{20})}{1-\frac{1}{3}}=2.9999999$ • The partial sums converge to 3.	» Circulate the room as discussion is taking place and offer assistance when required. » Write the student responses on the board.	» Do students recognise that no matter how big n is the partial will never exceed 3?

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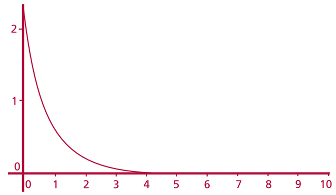
Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» We are now going to draw the function $f(x)=2(\frac{1}{3})^x$ Use the free software package available at www.geogebra.org if available. **Note:** x stands for the number of terms n and $f(x)$ is the graph that represents the terms of the series T_n. » What do you notice about this curve? » Does this agree with your theory that as n gets bigger, terms get successively smaller and eventually converge to zero? » What happens as $n \to \infty$? » We are now going to use *GeoGebra* to draw the function $f(x)=\dfrac{2(1-(\frac{1}{3})^x)}{(1-\frac{1}{3})}$ **Note:** x stands for the number of terms and $f(x)$ is the graph of the partial sums.	 • As x increases the $f(x)$ value approaches zero. • Yes • The terms get smaller and smaller.	» Use the free software package *GeoGebra* available at www.geogebra.org or another suitable functions graphing software application if available. **Note:** x stands for the number of terms and $f(x)$ is the graph that represents the terms of the series. » If using the software adjust the ratio of the x axis: y axis to enable students to see the approach to the limit more accurately. » Use *GeoGebra* to draw the function $f(x)=\dfrac{2(1-(\frac{1}{3})^x)}{(1-\frac{1}{3})}$	» Can students see that as n gets bigger the terms are approaching zero? » Do students understand in this case, $f(x)$ represents the partial sums and as n gets bigger it approaches 3?

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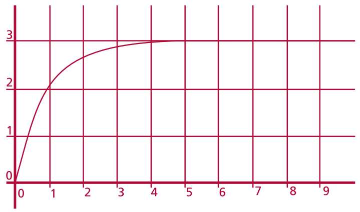
Teacher Reflections

» Student Learning Tasks: Teacher Input	» Student Activities: Possible and Expected Responses	» Teacher's Supports and Actions	» Checking Understanding
» What do you notice about this curve? » Does this agree with your theory that the partial sums of this series converges to 3 as $n \rightarrow \infty$?	 • As n gets bigger the function approaches 3. • Yes	» Discuss the graph and how in this case it converges towards 3.	
» When r the common ratio of a geometric series is a negative proper fraction do the partials sums of this series converge? Why? » When r the common ratio of a geometric series is a positive proper fraction do the partials sums of this series converge? Why? » When r the common ratio of a geometric series is a positive improper fraction do the partial sums of this series converge? » Do question 7 in **Section B. Student Activity 2.**	• Adding additional terms has negligible effect on the size of the partial sum as n gets bigger and hence converge. • As n gets bigger the value of the terms decrease and the effect of adding an extra term to the partial sum is insignificant and hence converge. • As n gets bigger the terms get bigger and the sum grows without limit and hence diverge.	» Write on the board $-1 < r < 0$ Convergent » Write on the board $0 < r < 1$ Convergent on the board. » Write on the board $\vert r \vert > 1$ Divergent. » Circulate the room and offer assistance as required.	» Do students understand that for the partial sums to converge r has to have a value between $-1 < r < 1$? » Do students recognise that of $-1 < r < 1$ then $\vert r \vert > 1$?

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
To develop the concept of the sum to infinity of a geometric series when -1< r <1			
» Now we are going to develop a formula for the sum to infinity of a geometric series when -1 < r < 1. That is when r is either a positive or negative proper fraction.			» Do students understand when the sum to infinity formula for a geometric series can be applied and how it differs from a geometric series?
» What is the formula for S_n?	• $S_n = \dfrac{a(1-r^n)}{1-r}$		
» Now we take the limit of this as n goes to infinity: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a(1-r^n)}{1-r}\right)$ $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a}{1-r}\right) - \lim\limits_{n\to\infty}\left(\dfrac{ar^n}{1-r}\right)$		» Develop the following on the board: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a(1-r^n)}{1-r}\right)$ $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a}{1-r}\right) - \lim\limits_{n\to\infty}\left(\dfrac{ar^n}{1-r}\right)$ But as $n \to \infty$ and r is in the range -1 < r < 1, r^n gets so small to be of no significance. Hence $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a(1-r^n)}{1-r}\right) = \left(\dfrac{a}{1-r}\right) = S_\infty$	
» What value did I say r has in this special case?	• -1 < r < 1		
» So as $n \to \infty$ what value has r^n?	• 0		
Hence $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\dfrac{a(1-r^n)}{1-r}\right) = \left(\dfrac{a}{1-r}\right)$ » This is the formula for the sum to infinity of a geometric series when 1 < r < 1.	• $S_\infty = \dfrac{a}{1-r}$		

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Supports and Actions	Checking Understanding
» Why does the sum to infinity formula for a geometric series only work when $-1 < x < 1$?	• Because in this case each consecutive term is getting so small to be less and less significant. When for example $r > 1$ (say 2) then the terms are getting successively bigger.		» Do students understand that the sum to infinity formula for a geometric series only applies when -1 < r < 1?
» Now do question 7 onwards in **Section B. Student Activity 2**.		» Some classes will need to be given an example of how the formula for the sum to infinity of a geometric series can be used to write a recurring decimal as a fraction.(See question 15 **Section B. Student Activity 2.**)	» Can students apply the formula to find the sum to infinity of a geometric series? » Do students understand what is meant by the Sum to infinity of a geometric series rather than just applying a formula?
» Working in pairs, devise your own questions involving the sum to infinity of a geometric series.		» If some students come up with sequences like 1 + 1/2 + 1/4 + 1/8 … and demonstrate that they understand the sum to infinity of a geometric series without real life applications that is sufficient.	» Do the questions devised by the students demonstrate an understanding of the concept of the sum to infinity of a geometric series?
Wrap up: Students now you are going to take the interactive quiz on geometric series on the student CD or available on line at http://www.projectmaths.ie/students/cd-strand3LC/strand3-number.asp		» A quick summary of all the material covered in this T&L should follow.	

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Teacher Reflections

Section A: Student Activity 1

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(Calculations must be shown in all cases.)

1. A person saves €1 in the first month and decides to double the amount he saves the next month. He continues this pattern of saving twice the amount he saved the previous month for 12 months how much will he save on the 12th month and how much will he have saved in total for the year ignoring any interest he received?
2. A culture of bacteria doubles every hour. If there are 100 bacteria at the beginning, how many bacteria will there be at the same time tomorrow, assuming none of these bacteria die in the meantime?
3. a. Find the sum of the first 5 powers of 3.

   b. Find the sum of the first 10 powers of 3.

   c. Find the sum of the second 5 powers of 3.
4. If the sum of the first 12 terms of a geometric series is 8190 and the common ratio is 2. Find the first term and the 20th term.
5. A person invests €100 and leaves it there for 5 years at 4% compound interest. Assume the interest is paid at the end of each year.

   a. What percentage of the original €100 will he have at the end of the first year?

   b. How much will he have at the end of the 1 year? Show your calculations.

   c. If you know what he had at the end of the first year, how will you work out what he had at the end of the second year?

   d. Is this a geometric sequence? If so what are "a" and "r"?

   e. How much money will this person have at the end of the 5 years?

   f. If the person in this question decided to invest €100 at the beginning of each year for 5 years. Find the total accumulated at the end of the 5 years.
6. Claire has a starting salary of €20,000 and gets a 5% increase per year. How much will she be earning in her 10th year in the job? How much in total will she earn in the first 10 years she works for this firm?
7. Find the sum of the 15th to the 32th terms (inclusive) of the series formed by the following geometric sequence 1, 3, 9, 27,...
8. Find the sum of the following geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\dots\frac{1}{4096}$.
9. The birth rate in an area is increasing by 6% per annum. If there are 20,000 children born in this area in 2009.

   a. How many children will be born in 2018 in this area?

   b. How many children in total will be born in this area between the years 2009 to 2018 inclusive?

Section A: Student Activity 1 (Continued)

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10. Alice saves €450 at the beginning of each year for 10 years at 5% compound interest added at the end of each year. How much will they have on the 10th anniversary of her first payment?

11. Alan aged 25 starts paying €2000 per annum into a pension fund and they are guaranteed a 4% interest rate throughout the savings period. How much will they have in the pension fund at the end of 30 years? Note for this particular pension the payments are made at the beginning of each year and the interest is paid at the end of each year.

12. A ball is bounced from a height of 2 metres and bounces back to ¾ of the height from which it fell. Find the total distance the ball travels, when it hits the ground for the 6th time to the nearest centimetre.
    Note the answer is not

$$\frac{200\left(1-\left(\frac{3}{4}\right)^{6}\right)}{1-\frac{3}{4}}$$

13. A carpenter is drilling a hole in a wall, which is a metre wide. In the first second he drills 30 centimetres and every second after that he drills ¾ the distance he drilled the previous second. How long will it be until he has completely drilled through the wall?

14. Show that $log_2 x$, $log_4 x$, $log_{16} x$, are three consecutive terms of a geometric sequence. Find the sum of the first ten terms of the series $log_2 x + log_4 x + log_{16} x + \ldots$

15. A child lives 200 metres from school. She walks 60 metres in the first minute and in each subsequent minute he walks 75% If the distance walked in the previous minute. Show that she takes between six and seven minutes to get to school.

16. If $P(1.04)^{-1} + P(1.04)^{-2} + P(1.04)^{-3} + K\ P(1.04)^{-20} = 300$, find the value of P, to the nearest whole number.

17. After how many terms will $1+\frac{1}{3}+\frac{1}{9}+K$ differ from 1.5 by less than 0.001?

Section B: Student Activity 2

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(Calculations must be shown in all cases.)

1. Find $\sum_{n=1}^{6} 3^{n-1}$
2. How does $\sum_{n=1}^{6} 3^{n-1}$ differ from $\sum_{n=0}^{6} 3^{n-1}$?
3. What is the common ratio of this sequence $\sum_{n=0}^{6} \left(\sqrt{3}\right)^{n}$?
4. Write the following series in sigma notation 1 - 3 + 9 - 27 + ... 531441.
5. Find the number of terms in $\sum_{n=6}^{16} \left(\frac{1}{4}\right)^{n}$ and calculate $\sum_{n=6}^{16} \left(\frac{1}{4}\right)^{n}$.
6. Find, correct to three significant figures, the least value of $k \in N$ such that $\sum_{r=1}^{25} k\left(1.032\right)^{1-r} = 1000.$
7. Examine the partial sums of the series shown below and determine if they converge to a particular value. Explain your reasoning.

   a. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

   b. 1 + 3 + 9 + ...
8. Find the sum to infinity of the geometric series $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$ and explain what information this number provides.
9. Find the sum to infinity of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ and explain what information this number provides.
10. The cost of laying a cable comprises a fixed amount of €3,000 for every 50m and a variable cost of €3,000 for the first 50m, €2,400 for the next 50m, and €1,920 for the next 50m and this geometric decrease continues.

    a. Calculate the total cost of laying a 300m long cable.

    b. Show that theoretically the total variable cost cannot exceed €15000.
11. The first term of a geometric series is 2 and the sum to infinity of this geometric sequence is 3 find the common ratio.
12. Find the sum to infinity of the geometric series 49 + 14 + 2 + ...
13. Evaluate $\sum_{i=1}^{10} \left[3\left(0.5\right)^{i-1}\right]$
14. Evaluate $\sum_{i=1}^{\infty} \left[3\left(0.5\right)^{i-1}\right]$

Section A: Student Activity 1 (Continued)

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15. Why can you not find the sum to infinity of the series 1 - 1 + 1 - 1 + 1 + …
    using the formula $S_\infty = \frac{a}{1-r} = \frac{1}{1-(-1)} = \frac{1}{2}$

16. Why is it not possible to find the sum to infinity of the geometric sequence
    1 + 2 + 4 + 8 + 16 + … ?

17. a. Do you agree that $.\dot{3} = .33333... = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...?$

    b. Find the sum to infinity of the geometric series shown in part a.

    c. If the above sum to infinity is equal to $.\dot{3}$, what does it tell us about $.\dot{3}$ in fraction format?

18. Using the sum to infinity formula for a geometric series write each of the following recurring decimals as a fraction.

    a. $.\dot{7}$

    b. $.\dot{4}$

    c. $1.\dot{6}$

    d. $2.\dot{4}\dot{2}$

    e. $2.4\dot{2}$

    f. $0.4\dot{2}\dot{1}$

    g. $3.\dot{0}\dot{2}$

19. Find the first 4 terms of the geometric series if the first term is 6 and the sum to infinity is 12.

20. The sum to infinity of a geometric series is $5\frac{1}{3}$ and the common ratio is $-\frac{1}{2}$. Determine the first term.

21. The sum to infinity of a geometric sequence is 8 and the first term is 2, find the common ratio and the 5^{th} term.

22. The 2^{nd} term of a geometric sequence is 9 and the 4^{th} term is 1. Find the sum to infinity.

Section A: Student Activity 1 (Continued)

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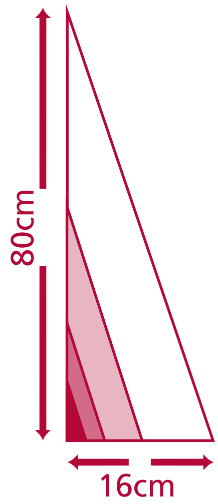
23. An art student starts with a white triangle that has a base of 16cm and a height of80 cm. Using different colours of construction paper the student places smaller triangles on top of the initial triangle. The side lengths of each triangle are exactly half the lengths of the previous triangle.



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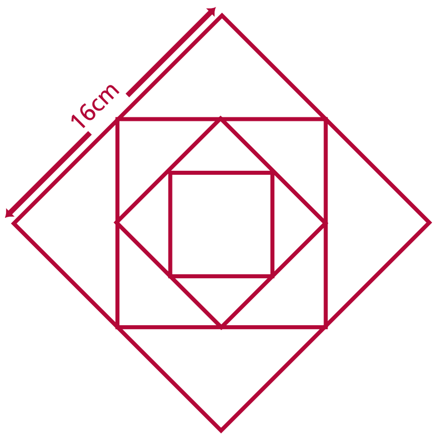
If in theory the student can continue this process an infinite number of times, determine the total area of all the triangles.

24. A square is drawn with a side length of 16cm. A second square is drawn using the midpoints of the first square, and has a side length of 8√2 cm. The process continues infinitely.



a. If the side lengths form a geometric sequence, determine the length of S_4.

b. The perimeter of the squares forms a geometric sequence. Find the perimeter of S_6 to the nearest tenth.

c. The areas of consecutive squares form a geometric sequence. Determine the common ratio of this sequence.

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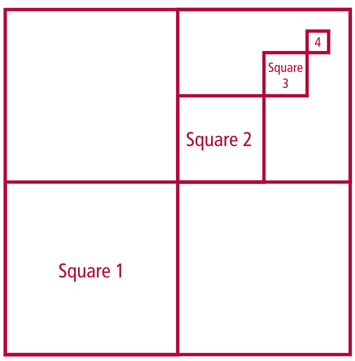
25. The area of square 1 is ¼ of the area of the square ABCD and the area of Square 2 is a ¼ of the area of square 1. Assuming this pattern continues for an infinite number of times, what fraction of the square ABCD will be covered by the coloured squares.



Appendix A

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To prove by induction the formula for the sum of the first n terms of a geometric series.

Aim:

To prove that S_n is equal to $\frac{a\left(1-r^n\right)}{1-r}$ for all geometric series, where S_n is the sum of the first n terms.

For n = 1:

$S_1 = \frac{a\left(1-r^1\right)}{1-r} = \frac{a\left(1-r\right)}{1-r} = a$ $\quad a = T_1$= The first term.

Hence it is true for n = 1.

For all n:

Assume true for $n = k$.

Hence $S_k = \frac{a\left(1-r^k\right)}{1-r}$

But $S_{k+1} = S_k + T_{k+1}$

Hence $S_{k+1} = \frac{a\left(1-r^k\right)}{1-r} + a\left(r\right)^{k+1-1}$

$$S_{k+1} = \frac{a\left(1-r^k\right)+ar^k\left(1-r\right)}{1-r}$$

$$S_{k+1} = \frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$$

$$S_{k+1} = \frac{a-ar^{k+1}}{1-r} = \frac{a\left(1-r^{k+1}\right)}{1-r}$$

This proves that $S_{k+1} = \frac{a\left(1-r^{k+1}\right)}{1-r}$.

Hence the formula is true for $n = k + 1$ provided it is true for $n = k$, but it is true for n = 1, hence it is true for all n.

Note: n is always a natural number.

