Project Maths
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Development Team

Module 3

JUNIOR CERTIFICATE

1. Probability Scale
2. Relative Frequency
3. Fundamental Principle of Counting
4. Outcomes of simple random processes
5. Basic set theory (HL)
6. Equally likely outcomes
7. Single Events questions
8. Multiple event questions
9. Tree Diagrams (HL)

LEAVING CERTIFICATE

1. All Junior Certificate Content
2. Arrangements and Selections
3. Set theory
4. Conditional Probability
5. Expected Value
6. Bernoulli Trials
7. Normal Distribution
8. Empirical Rule
9. Standard Normal Distribution (HL)
10. Standard Scores (z values) (HL)
11. Hypothesis Testing using Margin of Error (HL)

Concepts of Probability: Probability Scale

Example

The probability line shows the probability of 5 events A, B, C, D and E
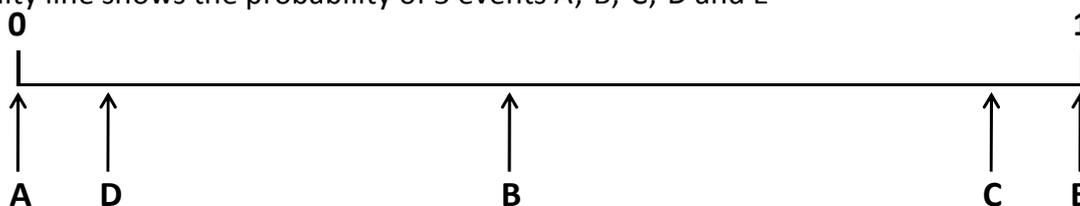
Which event is:

(a)   Certain to occur
Give an example of such an event

(b)   Unlikely but possible
Give an example of such an event

(c)   Impossible
Give an example of such an event

(d)   Likely but not certain
Give an example of such an event

(e)   Has a 50:50 chance of occuring
Give an example of such an event

Counting: Listing all Possible Outcomes of an Experiment

Example

A calculator can be used to generate random digits. Sandra generates 100 random digits with her calculator, she lists the results in the table below.

0							5									
1								6												
2							7											
3												8								
4												9													

Based on Sandra's results, estimate the probability that the calculator produces
(a) 9 **(b)** 2 **(c)** a digit that is a multiple of 3 **(d)** a digit that is a prime.
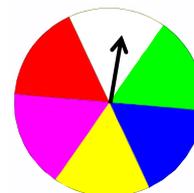
Solution:

$$\text{Estimated Probability} = \frac{\text{Number of successful trials}}{\text{Total number of trials}}$$

(a) $\dfrac{14}{100} = \dfrac{7}{50}$ **(b)** $\dfrac{6}{100} = \dfrac{3}{50}$ **(c)** $\dfrac{39}{100}$ **(d)** $\dfrac{40}{100} = \dfrac{2}{5}$

Outcomes of Simple Random Processes: Equally likely outcomes

Example

A spinner has 6 equal sectors coloured yellow, blue, green, white, red and purple.
After spinning the spinner, what is the probability of landing on each colour?

Solution

We have a one in six chance of landing on any of the colours. Each colour has an equally likely chance, The outcomes are all equally likely.

$$\text{Probability of an event} = \frac{\text{Number of successful outcomes}}{\text{Total number of outcomes}}$$

$$\text{Probability of each colour} = \frac{1}{6}$$

Fundamental Principle of Counting

> If one task can be accomplished in *x* different ways, and following this another task can be accomplished in *y* different ways, then the first task followed by the second can be accomplished in *x* by *y* different ways.

Example: A car manufacturer makes 3 models of cars, a mini, a saloon and an estate. These cars are all available in a choice of 5 colours; red, green, blue, brown and orange. How many different cars are available ?

Single Problem Events

Example A card is picked at random from a pack of 52 cards. What is the probability that it is **(i)** an ace **(ii)** a spade?

Solution **(i)** $\quad P(ace) = \dfrac{\text{Favourable Outcomes}}{\text{Total Outcomes}} = \dfrac{4}{52} = \dfrac{1}{13}$

(ii) $\quad P(spade) = \dfrac{\text{Favourable Outcomes}}{\text{Total Outcomes}} = \dfrac{13}{52} = \dfrac{1}{4}$
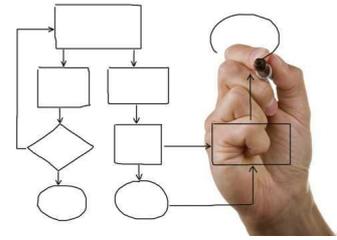
Example A bag contains 8 blue marbles and 6 white marbles. A marble is drawn at random from the bag. What is the probability that the marble is white?

Solution $\quad P(white) = \dfrac{\text{Favourable Outcomes}}{\text{Total Outcomes}} = \dfrac{6}{14} = \dfrac{3}{7}$

Drawing a Picture

It is often said that the three basic strategies in probability are:
1. Draw a picture.
2. Draw a picture.
3. Draw a picture.

Some of the ways that this can be done are as follows:
(a) Writing out the sample space
(b) Make a simple visual of the sample space such as a two way table
(c) Draw a tree diagram (HL Junior Cert.)
(d) Use a Venn diagram

Multiple Event Problems

Example   A die is thrown and a coin is tossed. What is the probability of getting a head and a 5?

Solution
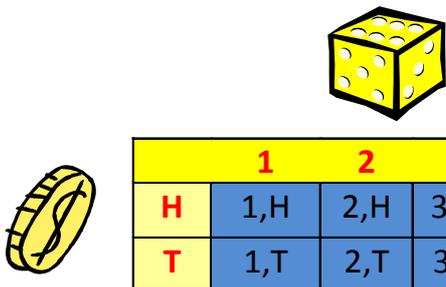
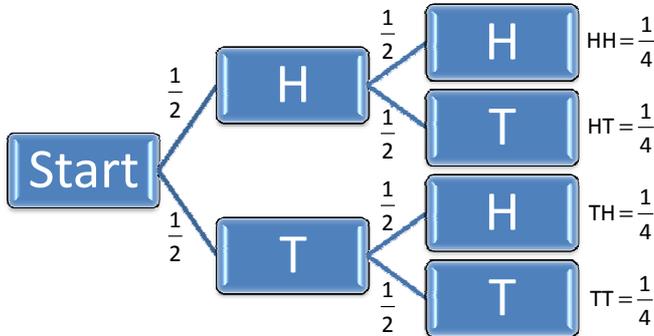|	1	2	3	4	5	6
H	1,H	2,H	3,H	4,H	5,H	6,H
T	1,T	2,T	3,T	4,T	5,T	6,T

$$P(\text{Head and } 5) = \frac{\text{Favourable Outcomes}}{\text{Total Outcomes}} = \frac{1}{12}$$

Tree Diagrams

> **Tree Diagram** are very useful for
> listing all possible outcomes (Sample Space).

Example   Draw a tree diagram to show the number of possible outcomes when two coins are tossed.

$$P(\text{Head and Head}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Example   Suppose that every child born has an equal chance of being a boy or a girl.

    (i)   Write out the sample space for the situation in which a mother has two children.

    (ii)   What is the probability that a randomly chosen mother of two childern will have two girls?

    (iii)   What is the probability that this mother of two children would have two boys?

    (iv)   What is the probability that this mother of two children would have one boy and one girl?

Solution   **(i)**
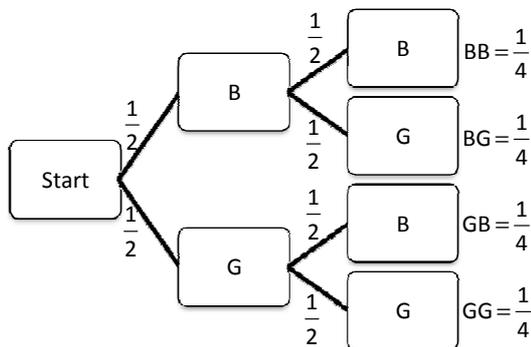
    (ii) $P(\text{G and G}) = \dfrac{1}{4}$

    (iii) $P(\text{B and B}) = \dfrac{1}{4}$

    (iv) $P(\text{1 B and 1 G}) = \dfrac{2}{4} = \dfrac{1}{2}$
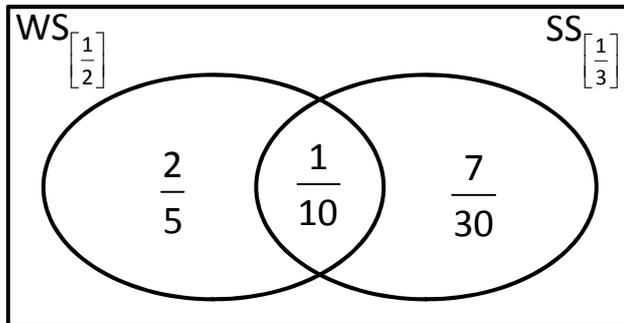
Probability and Sets

Example 1

In a class $\frac{1}{2}$ the students represent the school at Winter Sports and $\frac{1}{3}$ of the students represent the school at Summer Sports and $\frac{1}{10}$ at both.

Draw a Venn diagram to represent this.

If a student is chosen at random, what is the probability that someone who represents the school at sport will be selected?

Solution

WS$_{\left[\frac{1}{2}\right]}$ SS$_{\left[\frac{1}{3}\right]}$

$$\frac{2}{5} \qquad \frac{1}{10} \qquad \frac{7}{30}$$

P(Someone represents the school at sport) $= \dfrac{2}{5} + \dfrac{1}{10} + \dfrac{7}{30} = \dfrac{11}{15}$

$\left[\Rightarrow \dfrac{4}{15} \text{ is the probability than no one represents the school, which could be now entered on the venn diagram.} \right]$

Example 2

The probability that John will be on the school football team is 0.6. The probability that he will be on the school hurling team is 0.5 and the probability that he will be on both the football and hurling teams is 0.3. His father says that he will buy him a bicycle if he represents the school in any of the above sports. What is the chance that he gets the bicycle from his father?

Solution:

F$_{[0.6]}$ H$_{[0.5]}$

$$0.3 \qquad 0.3 \qquad 0.2$$

$$0.2$$
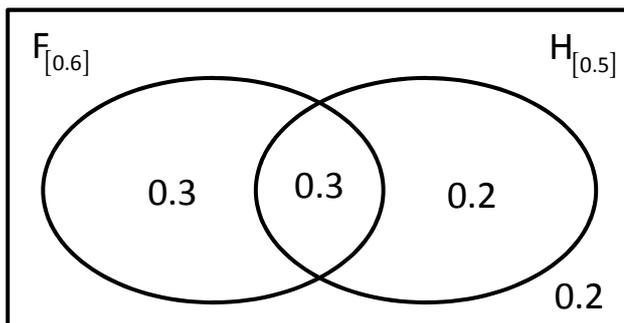
P(F or H) = 0.8

Example 3

There are 30 students in a class. 20 had been to France, 19 had been to Spain and 3 had been to neither country.

(a) Represent this on a Venn diagram and find how many students had been to both countries.

(b) If a student from the class is chosen at random, what is the probability that the student has:

    (i) visited France?

    (ii) visited France only?

    (iii) visited France or Spain?

    (iv) visited only one country?

Solution

$F_{[20]}$ $S_{[19]}$

8   12   7

3
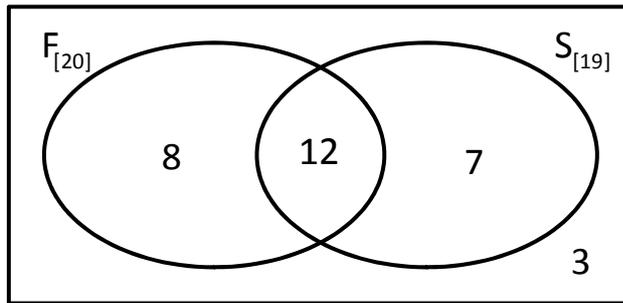
(i) $P(F) = \dfrac{20}{30} = \dfrac{2}{3}$

(ii) $P(F \text{ only}) = \dfrac{8}{30} = \dfrac{4}{15}$

(iii) $P(F \text{ or } S) = \dfrac{27}{30} = \dfrac{9}{10}$

(iv) $P(\text{one country only}) = \dfrac{15}{30} = \dfrac{1}{2}$

Permutations (Arrangements)

Example 1

In how many ways can we arrange the letters a, b, c, d, e, taking the letters two at a time?

Solution

Method 1 :   Write out the arrangements

     (a,b)   (a,c)   (a,d)   (a,e)   (b,c)   (b,d)   (b,e)   (c,d)   (c,e)   (d,e)

     (b,a)   (c,a)   (d,a)   (e,a)   (c,b)   (d,b)   (e,b)   (d,c)   (e,c)   (e,d)

     Total $= 20$ ways

Method 2 :   Box method

     $\square \times \square$

     $5 \times 4 = 20$ ways

Method 3 :   Use the P notation

     $^{5}P_{2} = \dfrac{5!}{(5-2)!} = \dfrac{5!}{3!} = \dfrac{5.4.3.2.1}{3.2.1} = 5.4 = 20 \text{ ways}$

Example 2

(i)   In how many ways can the letters of the word IRELAND be arranged if each letter is used exactly once in each arrangement?

(ii)   In how many of these arrangements do the vowels come together?

(iii)   In how many arrangements do the vowels not come together?

Solution

(i) $\boxed{}\times\boxed{}\times\boxed{}\times\boxed{}\times\boxed{}\times\boxed{}\times\boxed{}$

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! \text{ ways} = 5040 \text{ ways}$

(ii)   Treat the three vowels as one unit, but don't forget to move the vowels within the unit.

$\boxed{}\times\boxed{}\times\boxed{}\times\boxed{\text{I} \mid \text{E} \mid \text{A}}\times\boxed{}$

$5 \times 4 \times 3 \times \quad 2 \quad \times 1 \times 3! = 5! \times 3! = 120 \times 6 = 720 \text{ ways}$

(iii)   No. of ways in which the three vowels do not come together. This is a negative statement. This is equal to:

Total $-$ number of ways they come together.

$5040 - 720 = 4320 \text{ ways}$

Combinations (Selections)

Example 1

In how many ways can we select the letters a, b, c, d, e, taking the letters two at a time?

Solution

Method 1:   Write out the selections

(a,b)   (a,c)   (a,d)   (a,e)   (b,c)   (b,d)   (b,e)   (c,d)   (c,e)   (d,e)

Note: (a,b) is the same selection as (b,a)

Total $= 10$ ways

Method 2:   Use the C notation

Formal definition: $\binom{n}{r} = {}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

$\binom{5}{2} = {}^{5}C_{2} = \dfrac{5!}{2!(5-2)!} = \dfrac{5!}{2! \times 3!} = \dfrac{5.4.3.2.1}{2.1 \times 3.2.1} = \dfrac{5.4}{2.1} = \dfrac{20}{2} = 10 \text{ ways}$

Example 2

(i)   How many groups of five people can be selected from ten people?

(ii)   How many groups can be selected if two particular people from the ten cannot be in the same group?

Solution

(i) $^{10}C_5 = \dfrac{10.9.8.7.6}{1.2.3.4.5} = 252$ groups
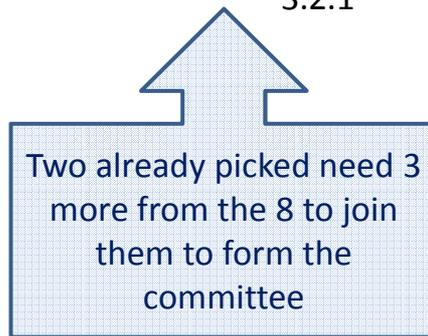
(ii)   This is a negative statement.

Total − number of ways the two people can be together.

Total = 252 groups

Number of ways the two people can be together = $^{8}C_3 = \dfrac{8.7.6}{3.2.1} = 56$

$252 - 56 = 196$ groups

Two already picked need 3 more from the 8 to join them to form the committee
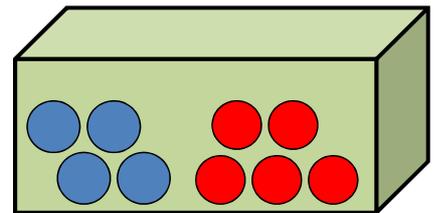

Probability Using the Counting Method

Example 1

A box contains four blue and five red discs. A disc is drawn at random from the box.

Find the probability that the disc is red.

Solution

$\text{Probability} = \dfrac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}}$

No. of favourable outcomes:
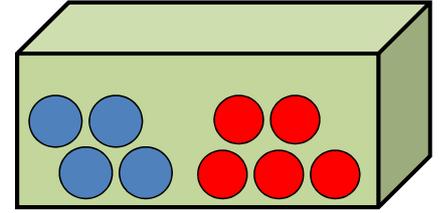
From 5 red choose $1 = {}^{5}C_1$

Total no. of outcomes:

From 9 discs choose $1 = {}^{9}C_1$

$\text{Probability} = \dfrac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{{}^{5}C_1}{{}^{9}C_1} = \dfrac{5}{9}$

Example 2

A box contains four blue and five red discs. Two discs are drawn at random from the box.

Find the probability that both discs are blue.

Solution

No. of favourable outcomes:

From 4 blue choose $2 = {}^4C_2$

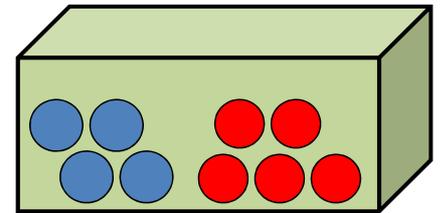Total no. of outcomes:

From 9 discs choose $2 = {}^9C_2$

$\text{Probability} = \dfrac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{{}^4C_2}{{}^9C_2} = \dfrac{6}{36} = \dfrac{1}{6}$

Example 3

A box contains four blue and five red discs. Two discs are drawn at random from the box.

Find the probability that exactly one red disc is drawn.

Solution

Note : This is the same as asking the probability of drawing one red disk and one blue disk.

No. of favourable outcomes:

From 5 red choose 1 and from 4 blue choose $1 = {}^5C_1 \times {}^4C_1$

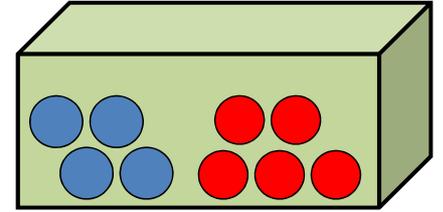Total no. of outcomes:

From 9 discs choose $2 = {}^9C_2$

$\text{Probability} = \dfrac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{{}^5C_1 \times {}^4C_1}{{}^9C_2} = \dfrac{20}{36} = \dfrac{5}{9}$

Example 4

A box contains four blue and five red discs. Four discs are drawn at random from the box.

Find the probability that exactly three of the discs are red.

Solution

No. of favourable outcomes:

From 5 red choose 3 and from 4 blue choose $1 = {}^5C_3 \times {}^4C_1$
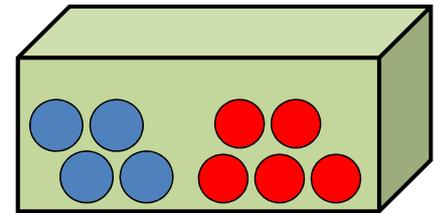

Total no. of outcomes:

From 9 discs choose $4 = {}^9C_4$

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{{}^5C_3 \times {}^4C_1}{{}^9C_4} = \frac{40}{126} = \frac{20}{63}$$

Example 5

A box contains four blue and five red discs. Three discs are drawn at random from the box.

Find the probability that at least one red disc is drawn.

Solution

Note :   This is the negative of drawing three blue disks

No. of Favourble outcomes = Total outcomes − unfavourable outcomes

No. of unfavourable outcomes:

From 4 blue choose $3 = {}^4C_3$

No. of favourble outcomes $= {}^9C_3 - {}^4C_3 = 84 - 4 = 80$

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{80}{{}^9C_3} = \frac{80}{84} = \frac{20}{21}$$

Replacement

Example 1

A card is picked from a pack of 52 and then replaced. Another card is then picked from the pack. What is the probability that the two cards picked are clubs?

Solution

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{^{13}C_1}{^{52}C_1} \times \frac{^{13}C_1}{^{52}C_1} = \frac{1}{16}$$

Example 2

Two cards are picked from a pack of 52. Find the probability that both cards are clubs.

Solution

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{^{13}C_2}{^{52}C_2} = \frac{12}{204} = \frac{1}{17}$$

Order

NOTE: By using the counting method you do not have to take order into account.

Example

Three cards are drawn from a pack of 52. Find the probability that the three cards drawn are the Jack of Hearts, the Queen of Diamonds and the King of Clubs.

Solution

Counting Method

Favourable outcomes:
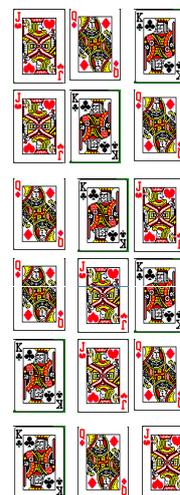
3 cards choose $3 = {}^{3}C_3$

Total outcomes:

52 cards choose $3 = {}^{52}C_3$

$$\text{Probability} = \frac{\text{no. of favourable outcomes}}{\text{total no. of outcomes}} = \frac{^{3}C_3}{^{52}C_3} = \frac{1}{22100}$$

Non Counting Method

$$\frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times 3! = \frac{1}{22100}$$

Lotto

Example

A lottery consists of drawing 6 balls and a bonus ball from a total of 45 balls. The balls are numbered from 1 to 45. If a person buys just one ticket:

(i) What is the probability of winning the lotto?

Solution

(i)

Jackpot €8,974,505

Winning Numbers   **Bonus**

Lotto (2) (6) (7) (9) (21) (39)   (45)

From 6 you want 6   **Not interested in bonus ball**

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{^6C_6}{^{45}C_6} = \frac{1}{8145060}$$

(ii) What is the probability of getting 3 of the winning numbers plus the bonus ball?

Jackpot €8,974,505

Winning Numbers   **Bonus**

Lotto (2) (6) (7) (9) (21) (39)   (45)

From 6 you want 3   **Want the bonus ball**   **From the remaining, non winning balls, you want 2**

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{^6C_3 \times {}^1C_1 \times {}^{38}C_2}{^{45}C_6} = \frac{14060}{8145060} = \frac{703}{407253}$$

(iii) What is the probability of getting 3 of the winning numbers?

Jackpot €8,974,505

Winning Numbers   **Bonus**

Lotto (2) (6) (7) (9) (21) (39)   (45)

From 6 you want 3   **Not interested in bonus ball**   **From the remaining, non winning balls, you want 3**

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total number of outcomes}} = \frac{^6C_3 \times {}^1C_0 \times {}^{38}C_3}{^{45}C_6} = \frac{168720}{8145060} = \frac{2812}{135751}$$