EXERCISE SET D

Q1 Let A and B be two nonempty sets where $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.
Consider each of the following relations:
$T = \{(1, a), (2, b), (2, c), (3, c), (4, b)\}$
$U = \{(1, a), (2, b), (4, b)\}$
$V = \{(1, a), (2, b), (3, c), (4, b)\}$
Which of these relations (T, U and V) qualify as functions?
Relation T, maps 2 ($\in A$) to both b and c ($\in B$). This violates condition 2 of the definition.
Relation T, is not a function.

Relation U is not defined for all elements of A.
This violates condition 1 of the definition. Relation U is not a function.

Relation V satisfies both conditions of the definition of a function.
Relation V is a function.

If we call the function f we have $f(1) = a, f(2) = b, f(3) = c$ and $f(4) = b$.

Q2
(i) Although the relation V in **Q1** is a function, it is not a one-to-one (or injective) function. Why?
(ii) V is an onto (surjective) function. Why?
(iii) Does the function f, defined by the relation V, have an inverse
(i) $f(2) = f(4) = b$, but $2 \neq 4$.
(ii) The range of f is equal to the set B (the codomain)
(iii) No, a function must be both injective and surjective to have an inverse.

Q3 For each of the relations $\{Q, R, S, T, U, V\}$ below, determine whether the relation is a function. If the relation is a function, determine whether the function is injective and/or surjective.

(i) $A = \{1, 2, 3\}, \quad B = \{a, b, c, d\}$
$Q = \{(1, a), (2, d), (3, b)\}$

(ii) $A = \{1, 2, 3\}, \quad B = \{a, b, c\}$
$R = \{(1, a), (2, b), (3, c)\}$

(iii) $A = \{1, 2, 3\}, \quad B = \{a, b, c\}$
$S = \{(1, a), (2, b), (3, b)\}$

(iv) $A = \{1, 2, 3\}, \quad B = \{a, b, c, d\}$
$T = \{(1, a), (2, b), (2, c), (3, d)\}$

(v) $A = \{1, 2, 3\}, \quad B = \{a, b\}$
$U = \{(1, a), (2, b), (3, b)\}$

(vi) $A = \{1, 2, 3\}, \quad B = \{a, b\}$
$V = \{(1, a), (2, b)\}$

(i) The relation is a function.
The function is injective.
The function is not surjective since c is not an element of the range.
(ii) The relation is a function.
The function is both injective and surjective.
(iii) The relation is a function.
The function is not injective since $f(2) = f(3)$ but $2 \neq 3$.
The function is not surjective since c is not an element of the range.
(iv) The relation is a not a function since the relation is not uniquely defined for 2.
(v) The relation is a function.
The function is not injective since $f(2) = f(3)$ but $2 \neq 3$.
The function is surjective.
(vi) The relation is a not a function since the relation is not defined for 2.

Q4 **(i)** Which of the relations in **Q3** is a bijection?
(ii) For the relation that is a bijection, write down the elements of the inverse function.
(i) Part **(ii)** $f = \{(1, a), (2, b), (3, c)\}$
(ii) $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

Q5. The function $\boldsymbol{f}$ is defined by: $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}: \boldsymbol{x} \mapsto \boldsymbol{x}^2 + 2$.
(i) Give an example to show that f is not injective.
(ii) Give an example to show that f is not surjective.
(i) $f(-1) = f(1) = 3$ but $-1 \neq 1$, therefore the function is not injective**.**
(ii) There is no real number, x such that $f(x) = 1$ therefore the function is not surjective.
Or the range of the function is $y \geq 2$. The range of the function is not $\mathbb{R}$ (the codomain) therefore the function is not surjective.

Q6. The function $\boldsymbol{f}$ is defined by: $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}: \boldsymbol{x} \mapsto \boldsymbol{x}^2 - 6\boldsymbol{x}$.
(i) Give an example to show that f is not injective.
(ii) Give an example to show that f is not surjective.
(i) $f(6) = f(0) = 0$ but $6 \neq 0$, therefore the function is not injective**.**
(ii) $f(x) = (x-3)^2 - 9$ [by completing the square]
There is no real number, x such that $f(x) = -10$ the function is not surjective.
Or the range of the function is $y \geq 2$. The range of the function is not $\mathbb{R}$ (the codomain) therefore the function is not surjective

Q7. For each of the functions below determine which of the properties hold, injective, surjective, bijective. Briefly explain your reasoning.
(i) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.
(ii) The function $f: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$.
(iii) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x + 1)x(x - 1)$.
(iv) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 - 9)(x^2 - 4)$.
(i) This function is injective, since e^x takes on each nonnegative real value for exactly one x. However, the function is not surjective, because e^x never takes on negative values. Therefore, the function is not bijective either.
(ii) The function e^x takes on every nonnegative value for exactly one x, so it is injective, surjective, and bijective.
(iii) This function is surjective, since it is continuous, it tends to $+\infty$ for large positive x, and tends to $-\infty$ for large negative x. The function takes on each real value for at least one x. However, this function is not injective, since it takes on the value 0 at $x = -1, x = 0$ and $x = 1$. Therefore, the function is not bijective either.
(iv) This function is not surjective, it tends to $+\infty$ for large positive x, and also tends to $+\infty$ for large negative x. Also this function is not injective, since it takes on the value 0 at $x = 3, x = -3, x = 4$ and $x = -4$. Therefore, the function is not bijective either.

EXERCISE SET E

Q1 **(i)** In each part state the natural domain and the range of the given function:

(a) $f(x) = x^2$ **(b)** $g(x) = \ln x$ **(c)** $h(x) = \sqrt{x}$ **(d)** $k(x) = \frac{1}{x}$

(i)	**(a)**	Domain	$x \in \mathbb{R}$	Range $\{y \in \mathbb{R} \mid y \geq 0\}$
	(b)	Domain	$\{x \in \mathbb{R} \mid x > 0\}$ $x \in \mathbb{R}^+$	Range $y \in \mathbb{R}$
	(c)	Domain	$\{x \in \mathbb{R} \mid x \geq 0\}$	Range $\{y \in \mathbb{R} \mid y \geq 0\}$
	(d)	Domain	$\{x \in \mathbb{R} \mid x \neq 0\}$	Range $\{y \in \mathbb{R} \mid y \neq 0\}$

(ii) In each part find the natural domain and the range of the given function:

(a) $f(x) = x^2 - 6x + 13$ **(b)** $g(x) = \ln(x + 2)$

(c) $h(x) = \sqrt{1 - x}$ **(d)** $k(x) = \frac{x+2}{x-3}$

(a) Domain $x \in \mathbb{R}$ Range $\{y \in \mathbb{R} | y \geq 4\}$
(b) Domain $\{x \in \mathbb{R} | x > -2\}$ Range $y \in \mathbb{R}$
(c) Domain $\{x \in \mathbb{R} | x \leq 1\}$ Range $\{y \in \mathbb{R} | y \geq 0\}$
(d) Domain $\{x \in \mathbb{R} | x \neq 3\}$ Range $\{y \in \mathbb{R} | y \neq 1\}$

(iii) For each of the functions in part **(ii)** that has an inverse, state the domain and range of the inverse function.

(a) No inverse
(b) Domain $x \in \mathbb{R}$ Range $\{y \in \mathbb{R} | y > -2\}$
(c) Domain $\{x \in \mathbb{R} | x \geq 0\}$ Range $\{y \in \mathbb{R} | y \leq 1\}$
(d) Domain $\{x \in \mathbb{R} | x \neq 1\}$ Range $\{y \in \mathbb{R} | y \neq 3\}$

EXERCISE SET F

Q1. Which of the following cubic functions have an inverse?
[Hint: Finding the derivative of the function may help!]
(i) $f(x) = x^3 - 6x^2 + 3x + 7$ **(ii)** $f(x) = -x^3 - 6x^2 - 13x + 4$
(iii) $f(x) = x^3 + 3x^2 + 4x + 3$ **(iv)** $f(x) = -x^3 + 3x^2 - x - 1$
All cubic functions are surjective by their nature. So we check injective by seeing if the function is always increasing or decreasing
(i) $f'(x) = 3(x - 2)^2 - 9$ this function is not always increasing or decreasing, hence not injective
(ii) $f'(x) = -[3(x + 2)^2 + 1]$ this function is always decreasing
(iii) $f'(x) = 3(x + 2)^2 + 1$ this function is always increasing
(iii) $f'(x) = -3(x - 1)^2 + 2$ this function is always increasing

Q2. A is the closed interval $[0, 5]$. That is, $A = \{x | 0 \leq x \leq 5, x \in \mathbb{R}\}$.
The function f is defined on A by:

$$f: A \to \mathbb{R}: x \mapsto x^3 - 5x^2 + 3x + 5.$$

(a) Find the maximum and minimum values of f.
Maximum $(5,20)$ Minimum $(3, -4)$
(b) State whether f is injective. Give a reason for your answer. [SEC S2014, Q5 P1]
f is not injective as it has a local maximum and minimum in the domain $[0, 5]$, so it cannot be strictly increasing or decreasing.

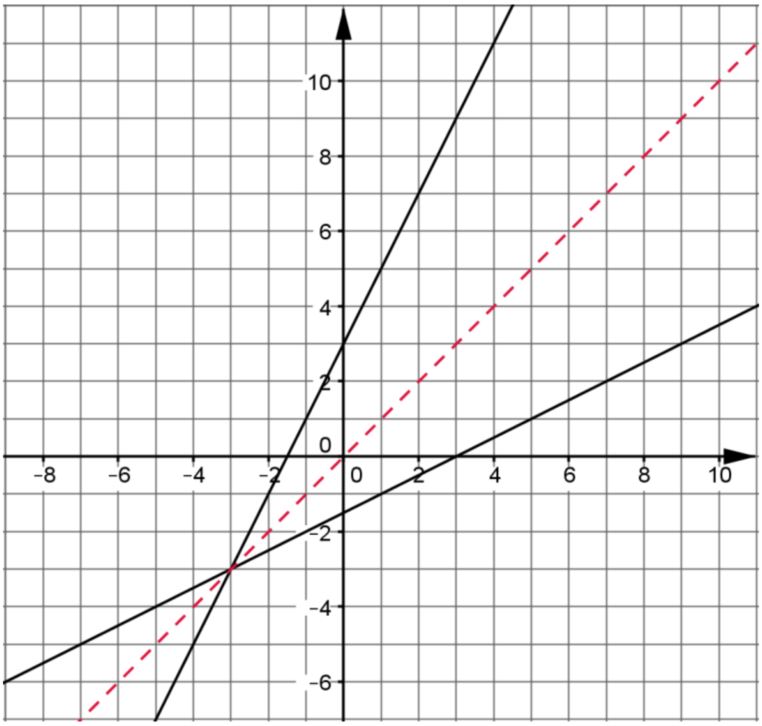
Q3. Consider $f: x \mapsto 2x + 3$.

(a) On the same axes, graph f and its inverse function f^{-1}, a reflection of f in the line $y = x$.

(b) Find $f^{-1}(x)$ using **(i)** coordinate geometry and the slope of $f^{-1}(x)$ from **(a)**

(ii) variable interchange.

(a)



(b)(i) f contains (0, 3) and (1, 5), therefore f^{-1} contains (3, 0) and (5, 1)

$$m = \frac{1-0}{5-3} = \frac{1}{2}$$

$$y = \frac{1}{2}(x-3)$$

$$\Rightarrow f(x) = \frac{1}{2}x - \frac{3}{2}$$

(ii) Swapping x and y.

$$y = 2x + 3$$

$$x = 2y + 3$$

$$2y = x - 3$$

$$f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$$

(c) Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

$(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$

$$= f\left(f^{-1}(x)\right) \qquad = f^{-1}\left(f(x)\right)$$

$$= f\left(\frac{1}{2}x - \frac{3}{2}\right) \qquad = f\left(2x+3\right)$$

$$= 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3 \qquad = \frac{1}{2}(2x+3) - \frac{3}{2}$$

$$= x - 3 + 3 \qquad = x + \frac{3}{2} - \frac{3}{2}$$

$$= x \qquad = x$$