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Using the context of saving money, prealgebra students learn algebraic operations without formal procedures.

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One of the basic disagreements in mathematics education concerns the roles that rules and procedures, on the one hand, and concepts and principles, on the other hand, should play in students' learning of mathematics. Regardless of the opinions expressed on this hotly debated issue, the use of procedures and an understanding of concepts are considered to be two separate aspects of mathematical activity.

Fortunately, an increasing number of mathematics educators have considered the two domains to be complementary rather than contradictory to each other. As a result, they recommend that the learning of rules and procedures be based on tasks that require a deeper understanding of meanings, a flexible choice of solution methods, and an ability to justify the use of the procedures used (Kieran 2004; Star 2005).

In this article, we present an application of these recommendations suitable for the classroom. Specifically, we describe an activity taken from a beginning algebra course and consider how students can learn procedures in a meaningful way.

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THE SAVINGS ACTIVITY

The Savings lesson explored the growth of different kinds of weekly savings plans. Our seventh-grade students completed the unit in two 90-minute sessions. Within the context of this unit, students learned the meaning of constants and variables using verbal, numerical, graphical, and symbolic representations, and by comparing the changes that took place.

In the first task, the change patterns of the weekly savings of four children were presented (see **fig. 1**). Each saving scheme was shown using a different representation. Students were then asked to compare the savings of the four children by choosing their favorite representation and using it to express all four saving schemes (see also Friedlander and Tabach 2001).

The second task focused on the procedure of adding linear expressions. The students were given information about the weekly savings of eight children, presented as algebraic expressions. They were required to combine amounts and consider the children in pairs (see **fig. 2**).

Our seventh-grade students worked on this activity during the third week of the school year, even though they had not yet learned how to add two symbolic expressions of the form $ax + b$. The students were expected, however, to understand that the expressions that describe the children's saving schemes were a shorter way of describing a change. For that purpose, they were expected to relate both to the initial (constant) amount and to the rate of change (the coefficient of x) of each expression. For example, the expression $30 + 12x$ shows the total savings for week x by combining an initial amount of \$30 and an additional weekly amount of \$12. (In the original problem, the term *shekels*—the national currency of Israel—was used.) Thus, the students were expected to base the meaning of algebraic expressions on the context of savings rather than follow rules of basic algebraic operations.

Fig. 1 The first task describes the savings of each person who is using a different representation.

Weekly Savings

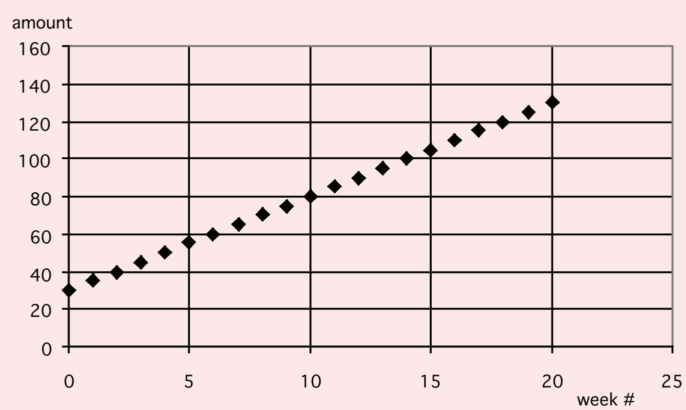
The savings of Diana, Yoni, Michael, and Danny changed during the last year, as described below. The numbers indicate amounts of money in dollars at the end of each week.

Diana: The table shows how much money Diana saved at the end of each week. The table continues in the same way for the rest of the year.

Week	1	2	3	4	5	6	. . .
Amount	\$7	\$14	\$21	\$28	\$35	\$42	. . .

Yoni: Yoni kept his savings at \$300 throughout the year.

Michael: The graph describes Michael's savings at the end of each of the first 20 weeks. The graph continues in the same way for the rest of the year.



Danny: Danny's savings can be described by the expression \$300 – 5*x*, in which *x* stands for the number of weeks.

STUDENT LEARNING

Michal Tabach taught the Savings activity in two seventh-grade beginning algebra classes. The lesson began with an initiation phase led by the teacher, followed by students working in pairs, and concluded with a whole-class summary discussion. Some illustrative examples of discussions from Tabach's lesson on adding linear expressions will follow.

The Initiation Phase

In this part of the lesson, the teacher presented the problem and asked such clarifying questions as these: "How can we describe in our own words Michael's savings? What does the expression representing Ben's savings, $-20 + 4x$, mean?"

Some students wondered what a negative sign meant when discussing savings. Others remarked that combining Danny's savings with any other child's savings would be unfair, since Danny could not increase his savings in any joint effort. The students' remarks and questions showed that they were interested and actively involved in the situation at hand. Next, the students worked on the task and added corresponding pairs of expressions.

The Students' Work Phase

The following two examples illustrate how students reasoned and used context-based meanings in learning an algebraic procedure.

Example 1. Natallie and Eve discussed the expressions for the combined savings of Diana ($7x$) and Karen ($10x$) and of Diana ($7x$) and Michael ($30 + 5x$):

Natallie: Diana gets 7 each week, and Karen gets 10 each week, so together they get 17 each week, which is $17x$.
Eve: They will get the largest amount of money.
Natallie: Diana and Michael . . . if Diana gets 7, and Michael gets 5, wait a minute! Michael has here 30, doesn't he? So $5 + 7 = 12$. . . . OK. . . . $30 + 12x$.

Fig. 2 The second task has students add linear equations.

Buying a Walkie-Talkie

The following expressions describe the savings of different children (x shows the number of the week).

Karen	$7x$	Yoni	300
Diana	$10x$	Rubin	$60 + 3x$
Michael	$30 + 5x$	Ben	$-20 + 4x$
Danny	$300 - 5x$	Alex	$-70 + 7x$

At the beginning of the year, Ben had a debt, and each week he added $4 to his box.

The children decided that to buy a walkie-talkie that costs $400, they would form pairs.

1. Find expressions, as simple as possible, to describe the amount of money in the combined saving boxes of the following pairs. (Try first to express the combined savings in words.)

Diana and Karen	________	Michael and Danny	________
Diana and Michael	________	Michael and Yoni	________
Diana and Danny	________	Michael and Rubin	________
Diana and Yoni	________	Michael and Ben	________
Diana and Rubin	________	Michael and Alex	________
Diana and Ben	________	Danny and Rubin	________
Diana and Alex	________	Danny and Ben	________

2. Which of these pairs (or possibly another pair) will be the first to collect the $400 needed to purchase the walkie-talkie?

These beginning algebra students were able to add two algebraic expressions correctly, specifically, $7x + 30 + 5x$. They did not resort to syntactic rules, since they had not yet been learned, and they also avoided the common mistake of adding constants to coefficients (Tirosh, Even, and Robinson 1998). In their past experience with algebra, Tabach's students investigated change on two occasions: the saving schemes in the first task of this activity and the change of area and perimeter in a sequence of growing squares in a previous activity. In our view, the correct and meaningful performance of these students in adding algebraic expressions was closely related to their work on these context-based activities.

Example 2. Ronnie and Carmel attempted to find a simplified expression for the combined savings of Diana ($7x$) and Danny ($300 - 5x$).

Ronnie: Danny gets 300 at the beginning of the year, and each week he gets 5 less. For her [Diana], it goes up by 7, meaning that it will go up only by 2.
Carmel: For Diana, it goes up by 7.
Ronnie: $300 - 5x$.
Carmel: Yes, I know.
Ronnie: His expression in parentheses plus hers.

Encouraging Meaningful Learning

The "Savings" activity focused on adding linear algebraic expressions in a context that encouraged meaningful learning. We will briefly discuss some benefits of learning procedures in this way.

Understanding the difference between changing and constant quantities. Because of the meanings attached to variables, coefficients, terms, and expressions, the students made very few commonly observed errors in adding algebraic expressions. They did not add variables and constants and did not "close" expressions to obtain a final result: For example, they did not consider $30 + 5x$ as being equivalent to $35x$. In general, a conceptual approach allows students to learn how to use procedures "flexibly, accurately, efficiently and appropriately" (Kilpatrick, Swafford, and Findell 2001, p. 116).

Providing points of reference. We believe that providing context-based meanings to algebraic objects, concepts, and operations should occur at the initial stages of learning algebra—instead of at more advanced stages as possible applications. A context-based approach to beginning algebra provides points of reference that students can review at a more advanced stage of learning. As a result, the use of contexts has the potential to bridge the gap between arithmetic and algebra, as well as between concrete and abstract objects. Contexts also enable students to learn algebraic procedures in a meaningful way (Tabach and Friedlander 2008).

Spontaneous evaluation of work. The need to relate their algebraic expressions to the context of saving money encouraged the students to frequently evaluate the correctness and the quality of their results. It also encouraged them to assess the progress made in the process of achieving their goal. For example, as described earlier, Ronnie and Carmel checked their expressions by substituting numbers and evaluated their results against the task requirements: "Good. But we need to find the shortest expression." We believe that a spontaneous evaluation of work (for example, checking answers, monitoring work, and discussing the quality of solutions) is more likely to occur in a context-based learning situation than in work involving symbolical algebra activities.

Following various solution paths. A context-based task has the potential to lead students to a variety of interpretations, based on students' personal experience, choice of representations, and levels of mathematical sophistication. The three solutions presented by our students at the summary phase showed the connection between context interpretations and mathematical solutions.

[They write $(300 - 5x) + 7x$, and check their expression for the first week by substituting 1.]

Ronnie: Good. But we need to find the shortest expression.
Carmel: I have an idea: 300 – . . . , like adding up these two, but, 300 – 2 times . . . , not 2 . . . , yes, two times. . . .
Ronnie: She gets 7, he drops 5. No, it is not correct. She goes up by 7, he goes down by 5. Did you get the same answer? 7 – 5 = 2!
Carmel: $300 - 2x$.

[They calculate using $x = 7$ and realize they did not obtain the expected result.]

Carmel: Let's stay with our first expression.
Ronnie: It is so long! [Calling the teacher] Do we have to find a short expression?

Students' remarks and questions showed that they were interested and actively involved in the situation at hand

Natallie and Eve's, as well as Ronnie and Carmel's, manipulation of symbolic expressions were motivated by their interpretation of the "Savings" activity. On two occasions, the students felt the need to check their symbolic expressions by numerical examples. When meaning and syntax contradicted each other, Ronnie and Carmel chose to relate to meaning and questioned the need for a better syntactic outcome. ("Do we have to find a short expression?") The teacher helped Ronnie and Carmel find a correct expression. After that, they were able to obtain correct answers for the following summary-phase activity.

The Summary Phase

This activity focused on the pair of students who would be the first to save $400 to buy a walkie-talkie. The following three answers were presented and discussed by the teacher, who wrote them on the board:

- Student A suggested as a rule of thumb to "look at the pair with the largest starting point of one member and the largest pace of increase of the other." As a result, he claimed that Yoni and Karen ($300 + 10x$) will be the first pair to reach the target "because Yoni starts with the largest amount, and Karen gets the largest amount each week."
- Student B suggested that "Danny and Yoni [$600 – 5x$] will get the money sooner, because at the beginning of the year, they already had $600." Several students objected to this answer, claiming that Danny's loss of $5 each week cannot be paid off if he uses all his money to buy the walkie-talkie in the first week. "You should look at what happens next—those who started at 600 will end up with a debt, which is unacceptable."
- Student C argued that "Yoni and Michael have $330 + 5x$, and they will be the first to get the money, since they had 330 at the beginning of the year and added 5 each week."

Each of the three answers resulted from sound mathematical considerations. Natallie's argument was the result of correctly comparing the fourteen pairs of children mentioned in the text of the task. Ronnie's answer was based on a comparison of all possible pairs, whereas Amos correctly implemented a reasonable (but incorrect) generalization.

CONCLUSIONS

In this case, different interpretations led to different solutions, and comparing solutions led to the need for additional interpretations of the problem, such as limiting a child's weekly balance to positive numbers. Thus, in addition to giving meaning to mathematical concepts or procedures, the context of the problem had an unexpected impact on the acceptance or rejection of a solution. We felt, however, that the benefits of working on context-based problems and the use of different solution methods outweighed the risks of students following solution paths that were not related to the mathematical target.

We also found that context-based learning increases student motivation and willingness to become engaged in the activity. The benefits that relate to students' learning of algebra in context outweigh the risks.

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