

PAYING ATTENTION TO ALGEBRAIC REASONING

K-12

Support Document for Paying Attention to Mathematics Education

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Paying Attention to Algebraic Reasoning

“Algebraic reasoning is a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways.”

(Kaput & Blanton, 2005, p. 99)

Algebraic reasoning is a gatekeeper for students in their efforts to progress in mathematics and science (Greenes et al., 2001). An early introduction gives all students more opportunities in later mathematics and career choices, and it may serve to support the transition to formal algebra in secondary grades, which research has shown to be difficult for most students (e.g., Kieran, 1992).

Paying Attention to Mathematics Education provided an overview of what it would take to help Ontario students make – and sustain – gains in their learning and understanding of mathematics. It outlined seven foundational principles for planning and implementing improvements and gave examples of what each principle would involve.

This document takes a more concrete approach and concentrates on the first principle: *Focus on mathematics*. *Paying Attention to Proportional Reasoning* was the first in this series of support documents; *Paying Attention to Algebraic Reasoning* is the second. Future support documents will explore other key topics in mathematics teaching and learning.

Algebraic reasoning connects the learning and teaching of arithmetic in elementary grades to functions and calculus in secondary grades. It provides a foundation for the development of abstract mathematical understanding. We hope this document serves to spark discussion and learning about this complex and important topic, both with colleagues and with students in your schools and classrooms.

SEVEN FOUNDATIONAL PRINCIPLES FOR IMPROVEMENT IN MATHEMATICS, K–12

- ❖ Focus on mathematics.
- ❖ Coordinate and strengthen mathematics leadership.
- ❖ Build understanding of effective mathematics instruction.
- ❖ Support collaborative professional learning.
- ❖ Design a responsive mathematics learning environment.
- ❖ Provide assessment and evaluation in mathematics.
- ❖ Facilitate access to mathematics learning resources.

Why Is Algebraic Reasoning Important?

Algebraic reasoning underpins all mathematical thinking, including arithmetic, because it allows us to explore the structure of mathematics. We now recognize the importance of including algebraic reasoning in mathematics instruction from a very young age so that powerful mathematical ideas are accessible to all students.

Everyone has the capacity to think algebraically because algebraic reasoning is essentially the way humans interact with the world. We look for patterns, pay attention to aspects of the pattern that are important, and then generalize from familiar to unfamiliar situations. Algebraic reasoning is present in many instances of our lives; for example, comparing which cellphone provider offers a better contract or determining times and distances when driving involves thinking algebraically. Algebraic reasoning is also part of many careers:

- ❖ Architects and construction experts use algebraic reasoning to design buildings and determine materials needed to build structures.
- ❖ Software developers use algebraic reasoning when creating codes.
- ❖ Bankers use algebra to figure out mortgage and interest rates.
- ❖ Scientists use algebra in almost every field.

What Is Algebraic Reasoning?

“Algebraic thinking or reasoning involves forming generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful symbol system, and exploring the concepts of pattern and functions.”

(Van de Walle, Karp, & Bay-Williams, 2011, p. 262)

Algebraic reasoning permeates all of mathematics and is about describing patterns of relationships among quantities – as opposed to arithmetic, which is carrying out calculations with known quantities. In its broadest sense, algebraic reasoning is about generalizing mathematical ideas and identifying mathematical structures.

Although formal symbolic algebra is introduced in the Ontario mathematics curriculum in the late junior grades, it should be fostered and nurtured right from Kindergarten. Algebraic reasoning is sometimes thought of as being only symbol manipulation and taught only in secondary grades. However, most educators agree that an emphasis should be placed on students developing algebraic understanding before they are introduced to formal symbolic representation and manipulation. For example, when students in primary grades notice that the order of numbers being added does not change the sum no matter which two numbers are being added, they are focusing on the structure of the relationship rather than the specific numbers that they are working with. This focus on the generalized property of addition (commutative property) is algebraic reasoning ($3 + 4 = 4 + 3$).

Algebraic reasoning is based on our ability to notice patterns and generalize from them. Algebra is the language that allows us to express these generalizations in a mathematical way. Formulas, such as the area of a rectangle = length \times width ($A = l \times w$), are derived through such generalizations. “Generalization

is the heartbeat of mathematics and appears in many forms. If teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place" (Mason, 1996, p. 65).

Let's consider some examples of generalizations.

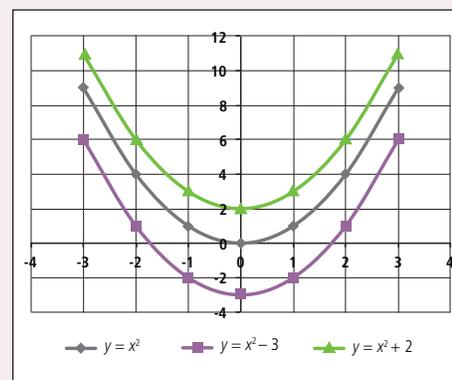
Adding zero to a number does not change the value of the number. A student might say, "I added zero to a bunch of numbers and it didn't change the number!" or "A number plus zero is the same number." These statements show a powerful understanding and are perfect examples of generalization. Algebra allows us to express the generalization in several ways, including concretely, pictorially, graphically, and symbolically ($n + 0 = n$). This algebraic expression represents the fact that no matter what quantity n stands for, adding zero will never change that quantity.

Many curriculum expectations engage students in exploring a relationship and developing generalizations to support their conceptual understanding of the relationship represented by a formula. It is important to recognize that the act of identifying the relationship and expressing it as a generalization represents significant learning for students. The formula is the outcome of students' algebraic reasoning and should not be the sole focus of the learning. For example, when students explore the perimeter of a rectangle, they can arrive in a number ways at the understanding that the perimeter is the sum of the sides. They may express this as perimeter = length + length + width + width, perimeter = $2 \times$ length + $2 \times$ width, or perimeter = length + width + length + width. These are all valid expressions and provide students with greater flexibility in recognizing when the generalization of $P = 2(l + w)$ should be used, as well as allowing them to develop and refine their ability to reason algebraically, understand the origin of the formula, and know how to adapt it for specific situations.

In secondary mathematics, students are reasoning algebraically when they identify the connections between algebraic and graphic representations of transformations of functions. Students may use technology to explore a number of graphs in order to understand that $y = x^2 + b$ is a vertical translation by b units of $y = x^2$ (Fig. 1).

Algebraic reasoning allows us to operate on any unknown quantity as if the quantity were known, in contrast to arithmetic reasoning, which involves operations on known quantities. The focus in algebra is on the *relationships* among quantities (which we call variables) and the ability to represent these different relationships.

Fig. 1 *Vertical Translations of Quadratic Functions*



Algebraic reasoning is important because it pushes students' understanding of mathematics beyond the result of specific calculations and the procedural application of formulas. Students require time to explore a variety of examples through which generalizations can be developed and applied flexibly to subsequent learning.

There are different approaches to developing algebraic reasoning. This document explores two significant approaches: generalizing arithmetic and functional thinking. Both methods are critical in the development of algebraic reasoning.

Algebraic Reasoning as Generalizing Arithmetic

Generalizing arithmetic is reasoning about operations and properties associated with numbers (Carpenter, Franke, & Levi, 2003). Generalizing arithmetic is about moving beyond calculations on specific numbers to thinking about the underlying mathematical structure of arithmetic by identifying the patterns found in arithmetic. Students can develop algebraic reasoning in several ways through generalized arithmetic. In this section, we look at the following:

- ❖ exploring properties and relationships
- ❖ exploring equality as a relationship between quantities
- ❖ using symbols, including letters, as variables

Exploring Properties and Relationships

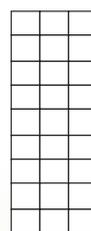
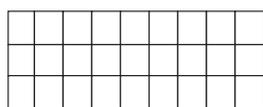
When reasoning algebraically, we explore properties and relationships of numbers, as well as operations on numbers. In an algebraic approach to arithmetic, it is important to consider arithmetic rules as general principles or properties. Rather than focusing on the results of specific calculations, for example, $2 + 2 = 4$, students can start to think about the properties of numbers. For example, think about the outcomes of adding different combinations of odd and even numbers: $\text{odd} + \text{odd} = \text{even}$, $\text{even} + \text{even} = \text{even}$, $\text{odd} + \text{even} = \text{odd}$. Rather than thinking about individual examples of adding odd and even numbers, students can think algebraically by noticing a pattern of results when adding odd and even numbers. When students can articulate this pattern and understand why it holds true for the sum of any pair of even and odd numbers, they have developed a generalization.

Other general properties of numbers that can be explored algebraically include being a prime number or a composite number and divisibility.

Students can also focus on the operations with whole numbers. Very young students can consider the commutative property of addition ($a + b = b + a$). A similar pattern can be found when multiplying two numbers ($a \times b = b \times a$). An array is a great tool for proving this property (Fig. 2).

Fig. 2 *Commutative Property of Multiplication* ($a \times b = b \times a$)

9 groups of 3 squares = 27 squares \longrightarrow 3 groups of 9 squares = 27 squares



Why is this important?

Rather than have students memorize properties or rules, allowing students to analyze many specific cases helps them move beyond thinking about multiple specific instances to thinking about the underlying mathematical generalization (Beatty & Bruce, 2012; Ontario Ministry of Education, 2005a). This enables students to substitute specific instances into the pattern; for example, $9 + 9 = 18$, $7 + 7 = 14$, $3 + 5 = 8$, with $\text{odd} + \text{odd} = \text{even}$ standing for every instance of adding two odd numbers to get an even sum. As well, an understanding that order does not affect the product of two numbers allows students to solve multiplication calculations more flexibly – for example, they might consider $4 \times \frac{1}{2}$ more readily as $\frac{1}{2} \times 4$ (one half of 4).

The focus of learning shifts from completing individual calculations to understanding the properties of and operations on numbers. This shift allows students to extend their understanding to other number systems (e.g., fractions, decimals, integers) and to algebraic expressions. Additionally, when secondary students are presented with more abstract algebraic concepts (e.g., $y = mx + b$), they will be able to connect to their earlier experiences with numbers while expanding on them in their new learning.

Exploring Equality as a Relationship between Quantities

It has been well documented that many students do not recognize that the equal sign denotes equality. As illustrated in the example below, most students see the equal sign as a signal to do something – to carry out a calculation and put the answer after the equal sign.

Teachers can do several things to establish = as a symbol of the relationship of equality. For example, once students are working with symbolic number sentences, they can consider true/false sentences (Carpenter, Franke, & Levi, 2003, p. 6). In this case, students are not asked to carry out calculations but instead to determine whether the number sentence is true or false. A sequence of true/false sentences presented to students might look like this:

$$3 + 4 = 7$$

$$5 + 1 = 7$$

$$7 = 3 + 4$$

$$4 + 3 = 5 + 2$$

Initially, most students will state that the first sentence is true. The second sentence may be deemed true if students are paying attention to the format or false if paying attention to the calculation. Students usually say that the third sentence is “backwards” because the expression is on the right: “How can you have the answer before you’ve done the adding?” And students typically consider the fourth to be nonsense at first. However, after discussing the meaning of the equal sign as indicating that the expressions on both sides are equal (or the same), most students can readily accept number sentences in which the “answer” does not immediately follow the equal sign.

Fig. 3 *Student Exploring Equality by Using a Balance Scale*



Balance scales are another approach to modelling problems such as $8 + 4 = _ + 5$ as they allow students to work on both sides of the equation. Even very young students can explore the concept of equality by using concrete materials and a balance scale without needing to see the symbolic number sentences.

Why is this important?

Algebra is about recognizing the relationships among quantities and operations. When students work with equations, it is imperative that they understand that the equal sign represents a relation between quantities rather than being a symbol to carry out a calculation. Students who develop this understanding can compare such problems as these

$$2 \times \text{___} + 15 = 31$$

$$2 \times \text{___} + 15 - 9 = 31 - 9$$

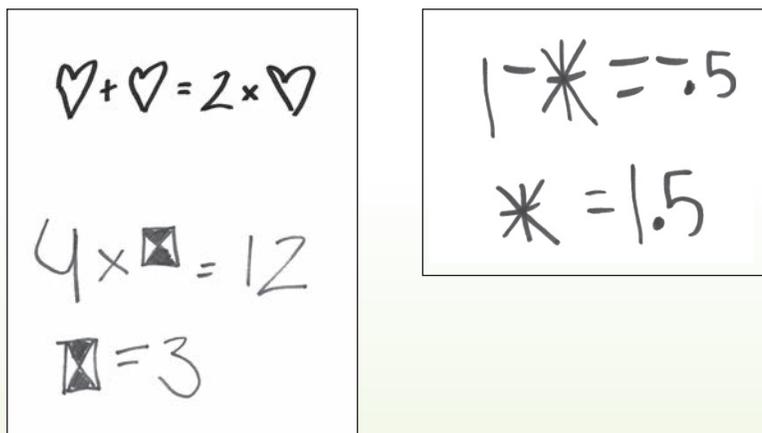
without having to carry out the calculations. They can focus on the equivalence of the two equations (algebraic reasoning) rather than comparing the two answers (arithmetic reasoning). Students who have developed an understanding of the equal sign will conclude that the value for ___ in both equations will be the same, since subtracting 9 from the expressions on both sides of the second equation does not change the relationship of equality between the two expressions.

Using Symbols, Including Letters, as Variables

At the heart of algebraic reasoning are generalizations as expressed by symbols. Traditionally, algebra was often presented in secondary school as a predetermined syntax of rules and symbolic language to be memorized by students. Students were expected to master the skills of symbolic manipulation before learning about the purpose and the use of these symbols. In other words, algebra was presented to students with limited opportunity for exploration or for meaning making. However, the Ontario mathematics curriculum specifies that students be provided with a range of opportunities for exploration and meaning making.

Early learners can start to use their own symbols (e.g., cat faces, hearts, trees) informally. Students can explore multiple cases of a property of arithmetic and then use symbols to make statements about the property (Fig. 4). Engaging in this kind of higher level thinking allows students to construct an understanding of the fundamental properties of addition, subtraction, multiplication, and division and of the relationships among operations.

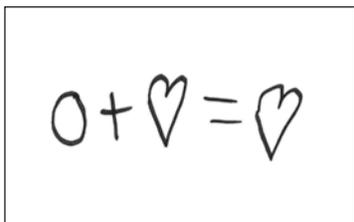
Fig. 4 *Early Learners Use Their Own Symbols*



As the learning progresses, students use symbols to model and make generalizations about increasingly complex mathematical systems (e.g., from linear to non-linear relations, real numbers to complex ones). The use of formal symbolic representations, such as equations, allows students to access these more abstract concepts. Eventually students will use letters (e.g., c , h , t) in place of the symbols (e.g., cat faces, hearts, trees). These letters can stand for a specific unknown value (variable), for example, $3 + c = 12$, or for a set of values, for example, $a + b = 12$.

Variables can also be used to represent the property of a number ($a + 0 = a$) (Fig. 5) or the underlying mathematical structure of a generalization (the sum of two consecutive numbers is always odd [$m + (m + 1) = 2m + 1$]). In these instances, m can be replaced with any number since the variable is a placeholder. One way to help students understand this use of variables is to ask, "Can you think of an open number sentence that is true no matter what numbers you put in for the variables?"

Fig. 5 *The Property of Adding Zero*



Using such sentences allows students to develop an understanding that letters can stand for more than one value, rather than one value in particular. It is important for teachers to help students understand the difference between these two uses of variables – those that represent a particular value ($x + 4 = 5$) and those that represent a range of values ($x + y = 5$).

Why is this important?

When students understand the multiple uses of variables, they are able to discern how they can manipulate the variable. As well, the particular variable does not make any difference as long as within a given problem the same variable has the same meaning. The value associated with a particular variable in one problem can be different in another problem. Because no connection exists between the variable and a specific value, it is common to use the same ones repeatedly: x , y , and z or a , b , and c . Using symbols rather than words allows us to whittle down verbose representations to deal with complicated expressions (e.g., $\frac{x+3}{x-5} + \frac{x+8}{x-3} = 17$) and more readily identify patterns within them.

Algebraic Reasoning as Functional Thinking

Functional thinking is analyzing patterns (numeric and geometric) to identify change and to recognize the relationship between two sets of numbers (Beatty & Bruce, 2012). This approach involves exploring how certain quantities relate to, or are changed or transformed into, other quantities.

Functional thinking is another form of generalizing. A function is a relation between two sets of data such that each element of a set is associated with a unique element of another set. Working with visual patterns (e.g., tiles, pictures) offers young students opportunities to think about relationships between quantities that go beyond calculations in arithmetic.

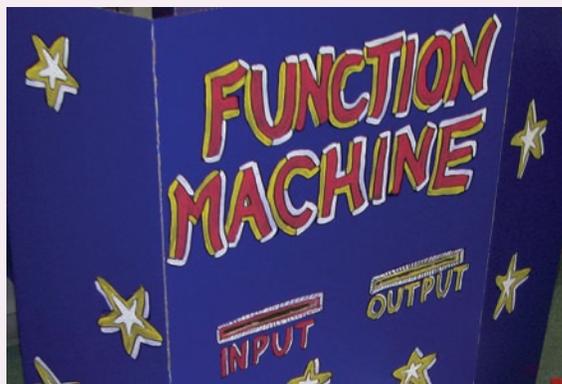
Students can develop functional thinking in several different ways. In this document, we explore the following:

- ❖ generalizing patterns
- ❖ using inverse operations

Generalizing Patterns

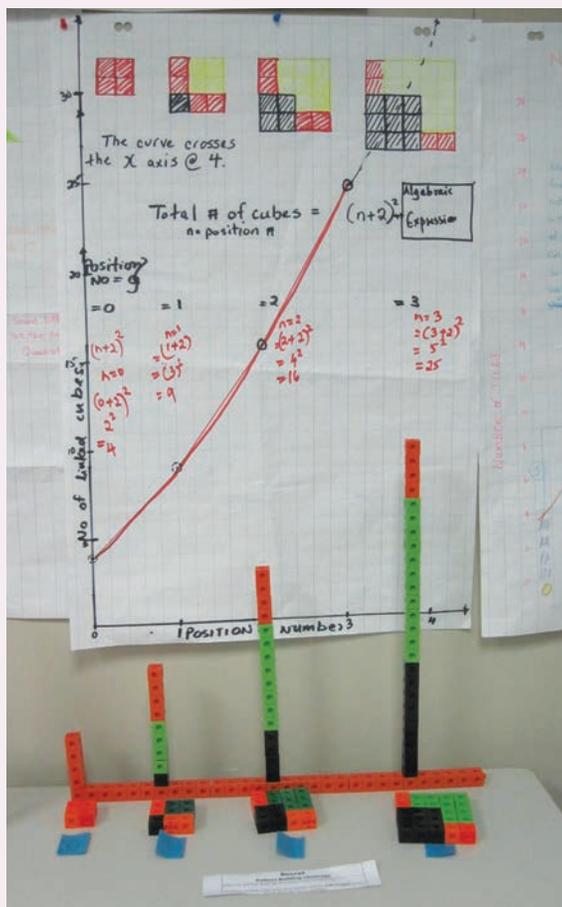
Some Ways to Explore Generalizing Patterns

Fig. 6 *Learning with a Function Machine*



Working with a function machine allows students to select input numbers that are then transformed by a particular rule to generate output numbers. This task engages them in thinking about the relationship that holds true for all input–output pairs.

Fig. 7 *Connecting the Visual to the Symbolic*



Older students connect visual representations with symbolic representations, allowing them to explore the properties of functions, including inverse operations.

Repeating patterns offer a way for very young students to find generalizations *within* a pattern, for example, by finding what comes next or what part repeats (recursive thinking). This type of thinking is foundational for developing an understanding of mathematical structure, and it supports the development of additive thinking.

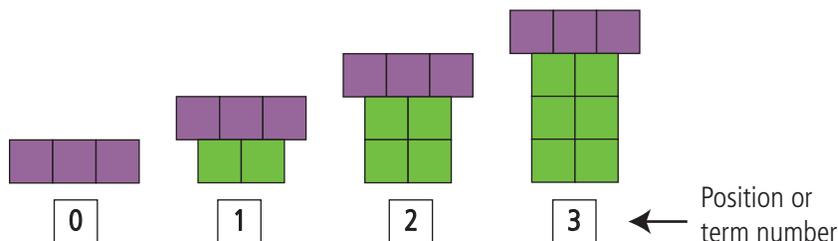
For example, in the repeating pattern below, the core comprises 2 green tiles and 1 purple tile, and the core repeats 4 times (Fig. 8). Recognizing this relationship allows students to accurately predict what comes next in the pattern.

Fig. 8 *Repeating Pattern*



Growing patterns offer a way for students to find generalizations *between* two sets of numbers. Working with growing patterns supports the development of multiplicative thinking and allows students to accurately predict values for any term of the pattern.

Fig. 9 *Growing Pattern*



Understanding the relationship between the number of tiles (dependent variable) and the position number (independent variable) allows students to predict the number of tiles required for any position in the pattern.

When encountering a growing pattern (Fig. 9), students may initially focus on the fact that the number of green tiles increases by two each time and might express this as “add 2 green tiles each time.” This kind of recursive thinking highlights the changes *within* one set of numbers, the tiles, but not on the relationship *between* the two sets of numbers – the position or term number of the pattern and the number of tiles at each position. Asking for the number of tiles at the 100th term highlights some limitations of this reasoning. Relying on recursion, “start with 3 tiles and add 2 tiles each time” or “add 2 tiles to the previous term,” makes it difficult to find a general rule that allows for the prediction of the number of tiles for *any* term of the pattern. Focusing on functional thinking emphasizes the relationship between the term number and the number of tiles at each term: as one set of numbers changes, the other set also changes in a predictable way; these two sets of numbers *co-vary*.

When students physically construct patterns and reflect on observed regularities, they will recognize this functional relationship. When constructing the pattern in Fig. 9, students may notice that they add 2 green tiles each time and that the 3 purple tiles stay in a row. Working with visual patterns allows students to identify which quantities vary (the position numbers and the green tiles) and which remain constant (the purple tiles) and the functional relationship. This functional relationship can be described as

multiply the position number by 2 (the tiles that increase) and add 3 more (the tiles that stay the same),

or

$$\text{number of tiles} = \text{position number} \times 2 + 3.$$

Students can also describe how the physical representation is connected to the rule, that is, the $+ 3$ comes from the purple tiles that stay the same (the constant), and the $\times 2$ comes from the green tiles that grow by 2 tiles for each consecutive position (the multiplier). Students can then generalize this understanding to other relationships, such as quadratic and exponential ones, and make meaningful connections between representations.

Why is this important?

Working with patterns can offer a way of engaging with mathematical quantities that goes beyond the “right” and “wrong” answers of arithmetic. Patterns can elicit multiple solution strategies from students, allow students to *think* about mathematical structure, and engage them in offering conjectures and justifying their thinking.

Using Inverse Operations

Students learn about the relationship between inverse operations through learning about fact families. Subtraction is the inverse operation of addition and vice versa. Division is the inverse operation of multiplication and vice versa. When thinking algebraically about a relationship between two numbers, we think of the first number as changing to become another number. For example, as well as thinking of $2 + 5 = 7$ as joining two parts (2 and 5) to make a whole (7), we can also think of it as adding 5 will change 2 into 7. The inverse operation, subtracting 5 from 7, takes us back to our original number (2).

We can illustrate the relationship of inverse operations by using unordered tables of values (Warren & Cooper, 2008) (Fig. 10). When the input numbers and change rule are given, students use addition to find the output (changed) values.

Fig. 10 *Unordered Table of Values Using Addition*

Input	Change Rule	Output
4	Add 5	9
2		
10		

Notice in Fig. 10 that the calculations follow the order of operations. In Fig. 11, we are undoing the change; thus, the operations are in reverse order.

When the output values and change rule are given, students use the inverse operation (subtraction) to find the input (original) values (Fig. 11).

Fig. 11 *Unordered Table of Values Using Subtraction*

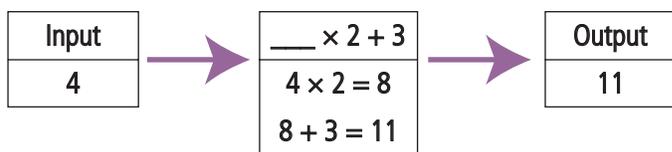
Input	Change Rule	Output
4	Add 5	9
		7
		10
		15

For further information on unordered table of values, see page 17.

The same is true for multiplication and division.

When working with composite rules, such as $input \times 2 + 3$ (the input number is multiplied by 2 then 3 is added), students learn how to undo two operations to find the original value. The thinking is reflected in Fig. 12 and Fig. 13.

Fig. 12 *Table Exploring Composite Rule When Input Is Provided*

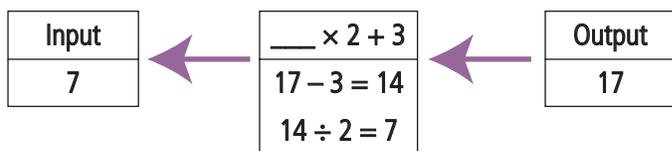


What is a composite rule?

A composite rule is a mathematical expression that has two functions; for example, a rule for a linear growing pattern can be a two-part rule that has both a multiplier and a constant.

(Beatty & Bruce, 2012, p. 190)

Fig. 13 *Table Exploring Composite Rule When Output Is Provided*



Why is this important?

When solving problems with unknown quantities, understanding inverse operations helps students in selecting and using appropriate strategies. Although many students have difficulty attaining this understanding, it supports subsequent mathematics learning. Take this example:

Dave gave Charlotte half of his hockey cards. Charlotte gave Johnnie half of the hockey cards she received from Dave. Johnnie kept 10 of those hockey cards and gave the remaining 8 to Dana. How many hockey cards did Dave give Charlotte?

Students may first connect the words to mathematical operations:

Dave gave Charlotte half $\div 2$

Charlotte gave Johnnie half $\div 2$

Johnnie kept 10 and gave the remaining 8 to Dana $- 10 - 8$

Students then work backwards:

$$8 + 10 = 18$$

$$18 \times 2 = 36$$

Dave gave Charlotte 36 cards.

Actions to Develop Algebraic Reasoning

The mathematical processes play a significant role in the learning and application of mathematics. In this section, we examine some specific actions through which students develop algebraic reasoning through the mathematical process of reasoning and proving as identified in the Ontario mathematics curriculum. The following actions are strongly interconnected and are integral to these processes:

- ❖ offering and testing conjectures
- ❖ justifying and proving
- ❖ predicting

What is a conjecture?

A conjecture is a guess or prediction based on limited evidence.

(Ontario Ministry of Education, 2005b, p. 60)

Offering and Testing Conjectures

Exploring algebraic relationships encourages higher order thinking. One of the most important aspects of developing algebraic thinking is helping students make conjectures about the properties of numbers and operations. It is important to record students' conjectures and have students refine their conjectures, which may be in the early stages of development and may be incorrect or imprecise. For example, a class working with examples like $12 - 12 = 0$ and $45 - 45 = 0$ initially came up with the idea that "if the first number is the same as the second number, the answer will always be zero." This was then refined to "If you subtract the same number from the same number, you will get zero."

Justifying and Proving

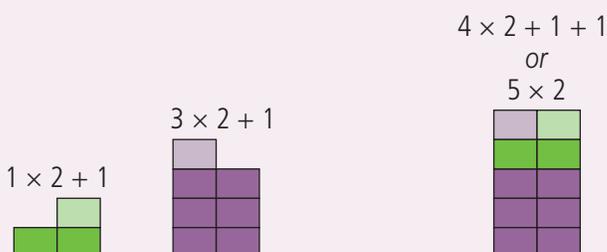
As important as generating conjectures is the practice of justifying or proving those conjectures. A good question for students to think about is, "Is this always true? How do you know?" Students are quick to understand that one counter-example – one time that the rule does not work – makes a conjecture false. They also come to understand that one definitive example does not prove a conjecture, and they may begin offering many examples where the rule works as proof. Students become more sophisticated in justifying proofs with practice. Fig. 14 offers an example of how a student thought through the concept of adding odd and even numbers.

Fig. 14 *An Example of a Student-Generated Proof for a Conjecture*

"An even number can always be divided into two equal groups. So all even numbers can be thought of as any number multiplied by 2, or $2n$. An odd number will always have one 'extra' in one of the two groups.

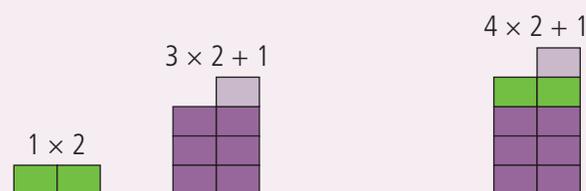
"So, all odd numbers can be thought of as any number multiplied by 2 with 1 added, or $2n + 1$. If you add even numbers together, you will have two even groups. It turns out that combining two odd numbers means you will also end up with two even groups. This is because adding two odd numbers means that you are combining two 'extras.'

"So if we break down $3 + 7$, we can think of it as



"We know if we combine two even numbers (1×2) and (3×2), we get an even number. If we add the two extras, which is $1 + 1$, or 2, we are adding an even number with an even number, which gives us two equal groups.

"But if you add an odd and an even number, one of the groups will have only one 'extra', so you end up with two equal groups plus 1."



Students can show their proof by using concrete tiles or pictures, as above, and they can use numbers and symbols to explain why an odd and odd number always sum to an even number:

$$\begin{aligned} (2x + 1) + (2y + 1) &= 2x + 2y + 1 + 1 \\ &= 2(x + y) + 2 \end{aligned}$$

Students could connect the evenness of the sum to the multiplier of 2 and the addend of 2.

Predicting

A pattern rule is a generalization that allows for the accurate prediction of specific values (the number of tiles needed for any position of the pattern or, conversely, the position number given any number of tiles). Students can go from "what comes *next*" to a *near* generalization (the number of tiles required for the 10th term of the pattern) to a *far* generalization (the number of tiles required for the 100th term of the pattern) to an explicit generalization (the number of tiles required for *any* position of the pattern).

It is the leap from *next* and *near* generalizations to *far* and *any* that constitutes the construction of an algebraic generalization. Asking students to predict unknown states, in this case unknown position numbers, prompts the need for a generalization that characterizes the data.

Making Connections among Representations

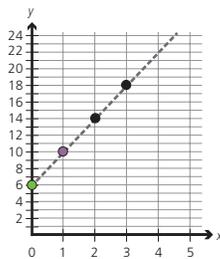
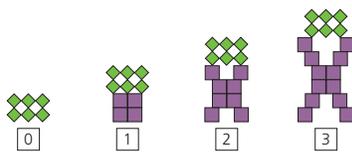
“Different representations of relationships (e.g. numeric, graphic, geometric, algebraic, verbal, concrete/pictorial) or patterns highlight different characteristics or behaviours and can serve different purposes.”

(Ontario Ministry of Education, 2009, p. 1)

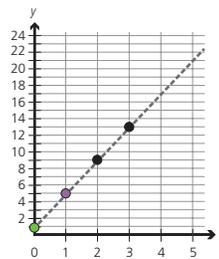
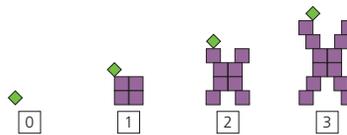
It is not just being able to use different representations that fosters students’ algebraic thinking; it is also being able to make connections among different representations. Students move through different phases, from using one representation, such as concrete or pictorial, to using more representations, such as graphs, word problems, and symbols. As they explore these different representations, students begin to make links between parallel representations and finally to integrate representations and flexibly move between them. For example, when students think flexibly, they are able to make predictions about one representation when changes are made to another. Fig. 15 demonstrates the impact of changing the rule on all the representations.

Fig. 15 *Exploring Changes to the Linear Growing Pattern Rule*

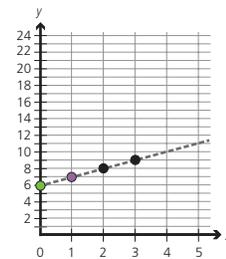
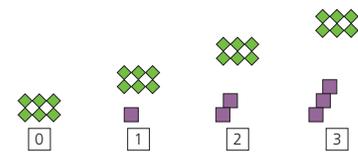
Suppose we have a rule,
 $y = 4x + 6$, a pattern, and a graph representing the rule.



How will changing the constant
in the rule affect the pattern or
the trend line on the graph?
The rule is now $y = 4x + 1$.



Or how would changing the
coefficient (multiplier) affect
the pattern or the trend line?
The rule is now $y = x + 6$.



Example from MathGAIns – CLIPS <http://www.mathclips.ca>

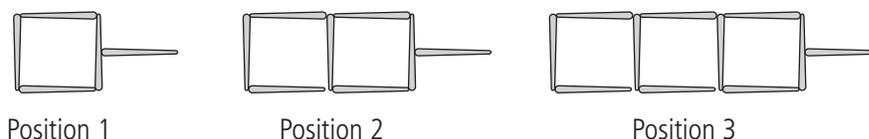
Algebraic Reasoning through Representation

Concrete or Pictorial Representations

Students in *all* grades should have the opportunity to work with visual representations repeatedly throughout their learning. It is important for students to initially work with concrete objects so that they can make sense of pictorial representations. Consider the linear growing pattern in Fig. 16. When the pattern itself is the site for problem solving, the actions of moving toothpicks (physically or mentally) allows students to determine the pattern, make near predictions (for position 4 or position 10) and far predictions (for position 100 or position 57), and then determine a generalized rule for any position number. This action allows students to explore the connection between two sets of data, in this case, the position number and the number of toothpicks at each position. By considering both the configuration and the number of toothpicks at each position, students can determine what part of the rule is responsible for the growth of the pattern (the multiplier $\times 3$) and what part is responsible for what stays the same (the constant $+ 2$).

Deconstructing visual representations, like concrete or pictorial patterns, is a more successful method of developing an understanding of generalizations than is memorizing rules for working with equations.

Fig. 16 *Example of a Growing Pattern*



Graphical Representations

Graphical representations are another kind of visual representation of an algebraic rule. The trend line on a graph is not just a collection of points but is itself a mathematical object that represents an algebraic rule. Graphs can indicate whether a rule is linear (Fig. 17) or quadratic (Fig. 18), or something else, and clearly show certain parameters of the rule – for example, in the linear graph in Fig. 17, the y -intercept of the graph represents the constant of a linear rule, and the slope represents the value of the multiplier.

Graphs also allow for the prediction of other values either by extrapolation (extending the trend line) or by interpolation (exploring the values between two points).

Graphical representations are another kind of visual representation of an algebraic rule.

Fig. 17 *Linear*

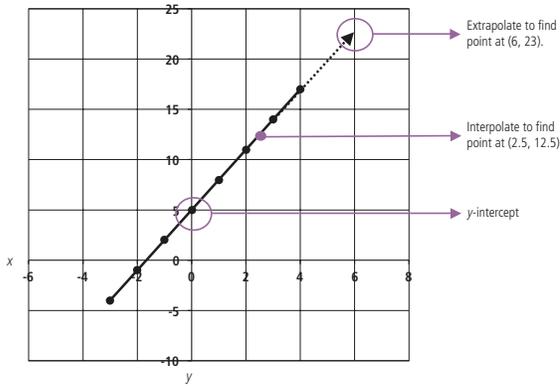
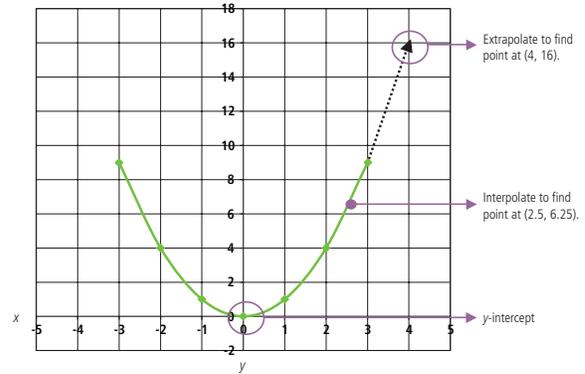


Fig. 18 *Quadratic*



Unordered and Ordered Tables of Values

Tables of values are useful for recording and presenting information

Fig. 19 *Unordered Table of Values*

Input (x)	Output (y)
2	11
5	20
1	8
3	14

$Input \times 3 + 5$

Fig. 20 *Ordered Table of Values*

Input (x)	Output (y)
1	8
2	11
3	14
4	17

$+ 3$
 $+ 3$
 $+ 3$

The assumption is that students can use these tables to look at the relationship between one column of values and the other. An unordered table of values ensures that students look across the columns to establish the relationship between the two sets of numbers, hence developing their functional thinking (Fig. 19). Presenting tables with ordered values (Fig. 20) can inhibit functional thinking (Watson, 2010).

In the ordered table (Fig. 20), students may simply look down the right-hand column to determine that the values in the column increase by 3 each time, which teaches them to take the first value in the *input* (x) column and multiply it by 3 to determine how many more are needed to get to the first value in the *output* (y) column ($1 \times 3 + 5 = 8$). In essence, students are merely using arithmetic to find the rule, which does little to help them understand that the rule represents a relationship across the two columns.

Using an ordered table of values can help students determine if a relationship is linear (Fig. 21) or quadratic (Fig. 22). First differences are the differences in y-values for consecutive x-values. If the differences are constant, then the relationship is linear. If the differences are not constant, students would look at second differences, which are the differences between consecutive first differences. If the second differences are constant, the relationship is quadratic.

Fig. 21 *Linear*

Input (x)	Output (y)
0	5
1	8
2	11
3	14
4	17

Fig. 22 *Quadratic*

Input (x)	Output (y)
0	0
1	1
2	4
3	9
4	16

Word Problems and Stories

Not surprisingly, when developing algebraic thinking, students initially perform better on word problems than on matched symbolic equations (Koedinger, Alibali, & Nathan, 2008). Verbal or written problems allow students to carry out quantitative reasoning based on their prior knowledge, without worrying about memorizing how to manipulate symbols.

Students correctly answered this:

Ziggy works as a waiter. He worked 5 hours in one day and got \$66 in tips. If he made \$111.90 that day, how much per hour does Ziggy make?

but struggled to solve this:

$$5x + 66 = 111.90$$

However, over time students develop an understanding of the connections between real-world contexts and their symbolic representations. This knowledge gives meaning to both the symbols and the procedures for solving equations.

Symbolic Representation

Students use formal representations and methods when they engage in carefully planned learning experiences designed to connect their prior conceptions of algebra with more formalized instruction. Focusing on multiple representations of algebraic reasoning before and during secondary school ensures that the introduction of symbols and prescribed rules for solving equations make sense. Students who are taught to rely solely on symbolic representations without having had an opportunity to explore other representations may develop an incomplete understanding of algebraic reasoning. Although students who have been taught how to use symbols and rules may be able to generate correct solutions, it is generally accepted that the learning of any mathematical procedure must be connected with conceptual knowledge to foster the development of deep understanding (Hiebert & Carpenter, 1992).

Algebraic Reasoning across Strands and Grades

Algebraic reasoning can be fostered through activities that encourage students to go beyond numeric reasoning to more general reasoning about relationships and quantities.

Here is an example of a question related to measurement that could be used with students from K to 12.

You want to build a rectangular garden. What are some possible gardens that have a perimeter of 24 metres? Record your results so that someone else can figure out your thinking.

1. What different rectangular shapes for the garden did you create?
2. How did you record your results?
3. How does the area of the garden change by making one side longer or shorter?
4. Have you found all the possibilities? How can you be sure?

Through the process of creating and analyzing different rectangles, students can generalize and see the relationships between the dimensions, the perimeter, and the area.

How Can We Promote Algebraic Reasoning?

“Problem solving provides students with opportunities to develop their ability to make generalizations and to deepen their understanding of the relationship between patterning and algebra.”

(Ontario Ministry of Education, 2005a, p. 9)

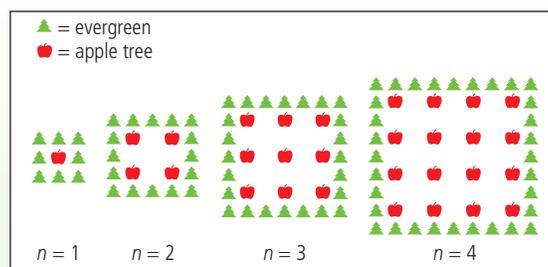
We can promote algebraic reasoning by creating a mathematical community that values conjecturing, justifying, predicting, and proving while problem solving. When teachers are thinking about the questions they will use, they should consider how the students are going to engage in the actions that develop algebraic reasoning. The following examples demonstrate how a simple change to a task can shift the focus from arithmetic thinking to algebraic thinking.

The following question engages students in generating a value for a specific term but not necessarily in thinking algebraically.

I noticed something interesting about my neighbours who farm apples. They plant their apple trees in square patterns in each orchard. To protect the trees from the wind, they plant evergreens all around the orchard.

The diagram to the right illustrates the pattern of apple trees and evergreens for any number (n) of rows of apple trees.

How many of each type of tree will there be when $n = 6$?



A student may respond as follows:

I drew a picture for position 5 and position 6 and then counted the number of evergreen trees and the number of apple trees. There are 48 evergreen and 36 apple trees in position 6.

Adapted from a PISA prompt

Slight adjustments to this question, as explored in the next section, will allow students to develop and refine their algebraic reasoning skills by generalizing patterns and relationships. This result is accomplished by extending an element of the task – for example, asking students to identify the relationship that allows an accurate prediction of the number of trees for any number of rows.

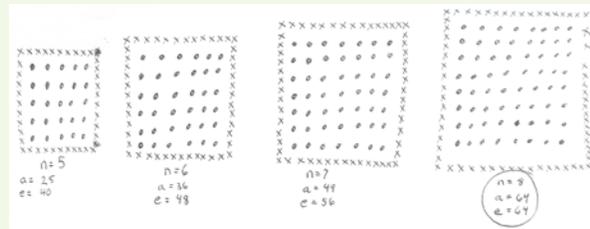
Being Responsive to Student Thinking

When educators examine and annotate student work by working with a partner or in small groups, it allows for student thinking to be unpacked – their understandings, strategies, and transitional conceptions. This is a powerful strategy for professional learning, which could be used in a school-wide or family of schools inquiry in which solutions are compared across grades.

To encourage algebraic reasoning, the prompt for the orchard problem was revised as follows:

1. When does the number of apple trees equal the number of evergreens? Justify your response.
2. How does the growth of the number of apple trees compare with the growth of the number of evergreens?

Ben created the subsequent terms as diagrams. After creating the diagram for the 8th term, and counting the number of evergreens and number of apple trees, he saw that the numbers of each tree were equivalent. Ben was not sure how to answer question 2.



To answer question 1, Kareem completed a table recursively by recognizing that the number of apple trees increases by 2 more than the previous increase. For example, between Term 1 and Term 2 the difference in the number of apple trees is 3. Between Term 2 and Term 3, the difference in the number of apple trees is 5; and between Term 3 and Term 4, the difference is 7. Using this pattern, he added 9 to 16 to generate the number of apple trees for Term 5. For the evergreens, Kareem noticed that the number of trees increases by 8 trees each time. To answer question 2, Kareem commented that the change in the number of evergreens is always 8, but the change in the number of apple trees increases (3, 5, 7, 9, 11, ...), so the number of apple trees increases at a more rapid rate after the first few terms.

Term	Apple	Evergreen
1	1) +3	8) +8
2	4) +5	16) +8
3	9) +7	24) +8
4	16) +9	32) +8
5	25) +11	40) +8
6	36) +13	48) +8
7	49) +15	56) +8
8	64) +15	64) +8

Jayda completed the table by using functional thinking and noticed that the number of apple trees is equal to the term number multiplied by itself (or squared). She also noticed that the number of evergreens is equal to the term number multiplied by 8. So, for term 8, the number of apple trees is equal to Term number 8^2 and the number of evergreens is equal to Term number 8×8 .

Term	Apple	Evergreen
1	1	8
2	4	16
3	9	24
4	16	32
5	25	40
6	36	48
7	49	56
8	64	64

apple = term #²
 evergreen = term # \times 8
 [so for the 8th term
 $8^2 = 64$ and
 $8 \times 8 = 64$]

To answer question 2, Jayda created another table to examine first and second differences and saw that the change in the number of evergreens is linear, but the change in the number of apple trees is quadratic. She then was able to identify that quadratic functions change at a more rapid rate over the subsequent terms of the pattern than linear relationships do.

Term	Apple	First Differences	Second Differences	Evergreen	First Differences
1	1	3		8	8
2	4	5	2	16	8
3	9	7	2	24	8
4	16	9	2	32	8
5	25	11	2	40	8
6	36	13	2	48	8
7	49	15	2	56	8
8	64			64	8

Chas generated two algebraic expressions: $a = t^2$ (where a represents the number of apple trees and t represents the term number) and $e = 8t$ (where e represents the number of evergreens and t represents the term number). These expressions represented the two patterns. He then equated the two expressions (when the number of apple trees is equal to the number of evergreens) and solved the equation. To answer question 2, Chas reasoned that multiplying a number by a constant value, such as 8, does not cause as much change as multiplying a number by itself (or squaring it) for larger numbers.

apple (t is term number) $\leftarrow t^2 = 8t \rightarrow$ evergreen

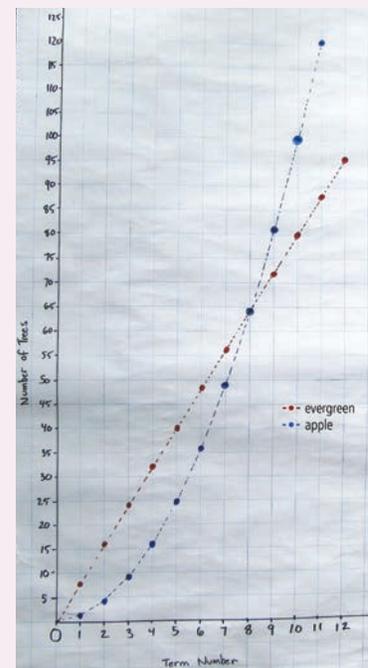
$$t^2 - 8t = 0$$

$$t(t-8) = 0$$

either $t=0$ or $t-8=0$
 $t=8$

If $t=0$ there are no trees!
 \therefore when $t=8$ there will be the same number of trees
 $8^2 = 64$ $8 \times 8 = 64$

When Lexie created a graph, she saw that the lines for each pattern intersect at the 8th term and knew that is the term number at which the number of evergreen trees and apple trees are the same. When examining her graph, Lexie commented that the change in the number of evergreens (red points) was constant because of the linear nature of the graph, whereas the change in the number of apple trees (blue points) increased at each term (since this graph is quadratic). She went on to compare the change in the number of evergreen trees with the number of apple trees before the point of intersection and after the point of intersection. She noticed that the number of evergreens grows at a more rapid rate than apple trees until the 8th term. After that point, the change in the number of apple trees is more rapid.



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Ministry Resources

Continuum & Connections: Patterning to Algebra K–3

A series of lessons provided as a sample of an instructional trajectory, designed to promote early stages of algebraic reasoning.

http://www.edugains.ca/resources/LearningMaterials/ContinuumConnection/PatterningtoAlgebraLessons_K-3.pdf

Guides to Effective Instruction in Mathematics Patterning and Algebra Grades K–3

<http://eworkshop.on.ca/edu/core.cfm>

Guides to Effective Instruction in Mathematics Patterning and Algebra, Grades 4–6

<http://eworkshop.on.ca/edu/core.cfm>

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<http://www.edugains.ca/newsite/math2/tips4rm.html>

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