Reflections on Practice

Project Maths
Tionscadal Mata
Development Team

Lesson Plan for [3rd Year, Polynomial factorisation**]**

For the lesson on [20/03/2015]
At [Presentation Secondary School, Wexford Town], [Deirdre Deegan's] class
Teacher: [Deirdre Deegan]
Lesson plan developed by: [Sean Rossiter, Mo O'Brien, Deirdre Deegan]

1. **Title of the Lesson:** Using arrays to factorise polynomials

2. **Brief description of the lesson**: Students will use the array model to help them factorise polynomials of degree 2 and degree 3.

3. **Aims of the Lesson:**

I'd like my students to develop their ability to work in a group and to use cooperative thinking in forming solutions.
I'd like students of all abilities to be engaged with and challenged by the lesson content.
I'd like students to explore the area model as a method for division of cubic expressions.
I'd like students to extend their understanding of the area model for its use in division of quadratic expressions.

4. **Learning Outcomes:**

As a result of studying this topic students will be able to:

- use the area model to divide quadratic expressions of the form $1x^2 + bx + c$ by linear expressions of the form $dx + e$ where b,c, d& e are positive integers.
- use the area model to divide quadratic expressions of the form $ax^2 + bx + c$ by linear expressions of the form $dx + e$ where a,b,c, d & e are integers.
- explain verbally the thinking needed to fill out the area model.
- explain verbally the need for a 3×2 array for division of a cubic expression by a linear expression.
- use the area model to divide cubic expressions of the form $1x^3 + bx^2 + cx + d$ by linear expressions of the form $ex + f$, where b, c, d, e & f are positive integers.
- use the area model to divide cubic expressions of the form a$x^3 + bx^2 + cx + d$ by linear expressions of the form $ex + f$, where a is a positive integer and b, c, d, e & f are positive integers.

5. **Background and Rationale**

The distributive property of multiplication is something that many students find easy but others find difficult to understand, to remember and to apply. The idea of distribution is encountered in first year, first with numbers and then with algebraic expressions. The ability to apply the distributive property of multiplication is important in all branches of maths and students lacking this ability often struggle to solve many basic maths problems. The array model has been demonstrated to be an effective tool for helping students to apply the distributive property of multiplication. Our students are presented with this method in first year, along with the more traditional approach of applying the distributive law. Students are not forced to use one method over another but are encouraged to choose the method which makes most sense to them.
One advantage of the area model is that it also offers an alternative approach to carrying out algebraic long division. Algebraic long division is a skill that many students find difficult to master. Even when the technique is presented alongside a numeric example (to allow students recognise the same process), many students still find it difficult to understand, to remember and to apply. This problem is exasperated by today's students being overly reliant on calculators from a young age. Because of this many of them have little experience of numeric long division in the first place.
In this lesson we hope to present students with an understandable and usable method for performing algebraic division using the area model. We hope that by giving students a solid understanding of multiplication using an area model that they will be able to apply some simple problem solving to carry out the reverse process of division.

6. **Research**

Under Section 4.6 of the Junior Certificate maths syllabus students are expected to:

1. multiply expressions of the form
    - $(ax+b)(cx+d)$
    - $(ax+b)(cx^2+dx+e)$ where $a,b,c,d,e\ \in Z$
2. divide expressions of the form
    - $ax^2+bx+c \div (dx+e)$
    - $ax^3+bx^2+cx+d\ \div (ex+f)$ where $a,b,c,d,e,\ f\ \in Z$
3. factorise expressions of the form
    - ax, axy where $a \in Z$
    - $abxy+ay$ where $a,b \in Z$
    - $sx-ty+tx-sy$ wheres,t,x,y are variable
    - ax^2+bx where $a,b \in Z$
    - x^2+bx+c where $b,c \in Z$
    - x^2-a^2
    - ax^2+bx+c where $a \in N \quad b,c \in Z$

7. About the Unit and the Lesson

1. Students will start by being asked to use the area model to multiply a pair of linear expressions. This task aims to remind students of the area model for multiplication of bracketed terms. While the students practice the technique, the teacher will highlight the important features of the area model which students need to understand before they can hope to use it for division. This activity covers the first learning outcome detailed in Section 6.
2. Students are then presented with a quadratic expression and one of its factors. They are asked to find the missing factor by working backwards using the area model. As students become comfortable with using the area model for division they are challenged to find the factors of more difficult quadratic expressions. This activity covers first part of the second learning outcome detailed in Section 6.
3. Students are presented with the division of a cubic expression by a linear expression. The first thing they are tasked with is determining the size of array needed. With teacher support they are then challenged to describe the process in a step-by-step way. To solidify students' learning they are asked to complete a matching activity based on division of a cubic expression by a linear expression. This activity covers the remaining part of the second learning outcome detailed in Section 6.

8. Flow of the Unit:

Lesson		**# of lesson periods**
	Evaluation of and operations on algebraic expressions - revision and extension of second year material	
1	• Terms, coefficients and expressions • Generating algebraic expressions from simple contexts • Evaluating expressions	2 x 40min.
2	• Adding and subtracting algebraic expressions • Multiplying terms and expressions, and using the associative and distributive properties to simplify expressions	1 x 40min.

3	• Dividing a quadratic expression by a linear expression • Dividing a cubic expression by a linear expression	2 x 40 min. **(of which the first lesson is the research lesson)**

9. Flow of the Lesson

Teaching Activity	Points of Consideration
1. Introduction The teacher explains what the aims of today's lesson are. 1. To factorise quadratic expressions. 2. To divide a cubic expression by a linear expression. The teacher explains that we are going to revise some important prior knowledge. Students are presented with the expression $(x+1)(x+2)$ and reminded that there are various ways of expanding the brackets. Students are asked to consider how to use the area model to expand the pair of brackets. The teacher asks individual students to describe how to use the area model to expand the bracket pair. The teacher asks students where each bracketed term should go on the diagram. The teacher asks students how to complete the four entries in the array. The teacher asks students how to use the four entries to write down their answer in the form: ax^2+bx+c. The teacher highlights the fact that when the array is used in this way, like terms end up along one of the diagonals, the highest order term ends up in the first space and the number term ends up in the final space.	The teacher writes the learning outcomes on the board for students to see. The teacher writes the quadratic expression on the board. The teacher projects an empty array on the board. The teacher fills in the different parts of the array as students describe the process. Can students describe how to use the array correctly? The teacher circles the like terms on the diagonal. The teacher writes out the answer to the question in the form: ax^2+bx+c.
2. Posing the Task The teacher writes a new problem on the board: $(x^2+6x+8)\div(x+2)$ and asks students if it would be possible to use the area model to answer this question. The teacher asks individual students to describe how the array could be set up to complete this division. With the support of students, he teacher demonstrates how the array can be completed to find the quotient.	The teacher may support students in their thinking by reminding them that division is the inverse process of multiplication. The teacher gives students some time to discuss how this could work. It is important to support and encourage students who are giving feedback to the class. Do students understand where the divisor is placed in the array? Do students understand that the dividend is placed in the four spaces in the array? Can students communicate that the highest order term in the dividend will go in the first space of the array while the number term will go in the last space? Do students understand that the x term of the dividend is split across the two remaining

	spaces?
3. Anticipated Student Responses Students should have little difficulty identifying where the divisor is placed in the array. Students may have difficulty understanding that the dividend must go in the four empty spaces in the array. Students should have little difficulty identifying that the x^2 term of the dividend should go in the first space of the array. Students may have difficulty identifying how to use the array to find the quotient. Students may struggle with the idea of using the sum of the x terms on the diagonal to complete the quotient. Students may not see that they can check their answer using the remaining entry in the array. Students are directed to complete Question 1 – 3 on their worksheet individually. They are told to compare their answers to those of their partner's and to check each solution if differences exist.	The teacher may need to lead the discussion so that students think about the important aspects of using an array to divide two expressions. The teacher may need to ask students specific questions such as "What must go in the first space of the array?" followed by "What does this mean for the first term in our answer?", followed by "What other information can we now fill into the array?" and so on. The teacher should highlight the fact that the array model can be used to check that the answer is correct. The teacher distributes copies of the worksheet to each student. The teacher circulates the room to check that all students are able to complete the worksheet. Students who finish quickly are encouraged to attempt more difficult examples.
4. Comparing and Discussing The teacher takes each question from the worksheet and asks students to hold up their answers on their show-me boards. The teacher asks individual students to describe how they used the array to answer each question.	Did all students complete the worksheet? Did all students answer all questions correctly? Can students describe the process they used to answer these questions? The teacher writes up each student's description on the board. The teacher emphasizes the location of the divisor, dividend and quotient and the existence of like terms along one diagonal.
5. Posing the task The teacher presents students with another division problem: $(x^2 - 2x - 15) \div (x - 5)$ and asks them to attempt solving it on their show-me boards. After some time the teacher asks students to hold up their answers. The teacher asks individual students to describe each step of the process of dividing the quadratic expression by the linear expression and writes up the solution on the board. The teacher asks students to complete Question 4 to Question 6 of their worksheet, again working individually and then checking their answers with their partner.	Can students apply their prior knowledge to solve this problem? Can students split the $-2x$ term correctly along the diagonal? The teacher circulates around the room to help students who are having difficulties with the task.
6. Anticipated Student Responses Students correctly use the area model to complete the division exercises.	The teacher needs to support students who have difficulties with integer operations. Students who complete the work quickly may

Some students my find using integers difficult due to not knowing how to correctly add/subtract integers or multiply integers.	attempt more challenging questions.
7. The teacher takes each question on the worksheet and asks students to hold up their answers on their show-me boards. The teacher asks individual students to explain how they solved each question using the area model. The teacher asks students why this activity was harder than the last.	If students have difficulties with specific questions the teacher may write the solution on the board. Do students understand that the presence of integers in the divisor and dividend make these questions more difficult? Do students recognize that difficulty with integers may hinder their progress in other areas of maths?
8. Posing the task The teacher writes the following division on the board: $(x^3 + 5x^2 + 7x + 3) \div (x + 3)$ and asks students what is different about this problem compared to the others they have done. The teacher asks students to draw the outline of the array needed to solve this problem. The teacher asks students to hold up their show-me boards. The teacher asks students why they need a bigger array this time? The teacher asks students to fill in as much information on their array. The teacher explains the process of dividing a cubic expression by a linear expression, questioning individual students to explain different steps in the process. The teacher asks students if there is a way in which they can check their answer. The teacher presents another division on the board: $(x^3 - 6x^2 + 3x + 10) \div (x - 2)$ and asks students to solve this on their show-me boards. The teacher asks students to show their solutions on their show-me boards. The teacher asks individual students to explain how they solved this problem. The teacher distributes a matching activity to each group of four students. Students are instructed to solve the four division problems in the activity by placing the various terms in the correct spaces in each array.	Do students recognize that they are now dividing a cubic expression by a linear expression? Do students understand that they need a 2×3 array? Can students explain why they need a 2×3 array? Do students understand that when they divide a cubic expression by a linear expression they will get a quadratic expression? Can students fill in some information on the array? Do students understand that one diagonal will hold the x^2 terms while another diagonal will hold the x terms? Can students split the x^2 terms and the x terms correctly to allow them to complete the array? Do students recognize that by filling in the final entry of the array they can check their answer? Can students apply the area model to work with a cubic expression with integer coefficients? The teacher circulates the room to observe student interaction and to question students about their reasoning. Can students apply their knowledge of the area model to complete the matching activity? Do students work well together as a team? Do students check each other's work? Do students speak up when they think that part of the group's work is incorrect? Do students explain their reasoning as they complete the matching activity?

	The teacher may suggest that students circle the diagonal for the x^2 terms and the diagonal for the x terms.
9. Anticipated Student Responses Students may find it difficult to imagine the array needed to complete this problem. Students may find powers of three more difficult to deal with. Students may struggle with matching the x^2 terms along one diagonal and the x terms on another diagonal. Students may find the matching activity too easy. Students may get confused by multiplication and addition. Students may have difficulties with negative integers.	
10. Comparing and Discussing The teacher asks students to hold up their completed work and commends the class for their exceptional efforts.	Students who have completed the task should have all correct answers due to the fact that it's a matching activity.
11. Summing up The teacher asks students what they have learned today and reinforces the learning outcomes written up on the board at the start of class. Students are asked to complete the remaining questions on their worksheet for homework.	

10. Evaluation

There will be three observers in the lesson along with the teacher.
The teacher is expected to interact with students during group works and to use suitable questioning strategies to find out about students' thinking. The teacher will use show-me boards to identify which students are able to complete the various activities and which students are having difficulties.
Observer 1will record the lesson flow and how each part of the lesson progresses in relation to the predicted timing.
Observer 2 will record which parts of the lesson students found difficult and which parts students found easy. Observer 2 will record what difficulties students had and how these difficulties were addressed.
Observer 3 will record examples of student work by photograph.
All observers will record details of student interaction including how students worked together as a group, if certain students were disengaged with the lesson, if students demonstrated initiative when finished a task and so on.
Students will be issued with a post-lesson questionnaire.
Teachers will have a post-lesson meeting to reflect on their observations from the lesson.

11. Board Plan

Aims: To factorise quadratic expressions
To divide a cubic expression by a linear expression

$(x^2 + 6x + 8) \div (x + 2)$

Dividing a quadratic by a linear

	x	$+4$
x	x^2	$+4x$
+2	$+2x$	$+8$

$(x^3 + 5x^2 + 7x + 3) \div (x + 3)$

Dividing a cubic by a linear

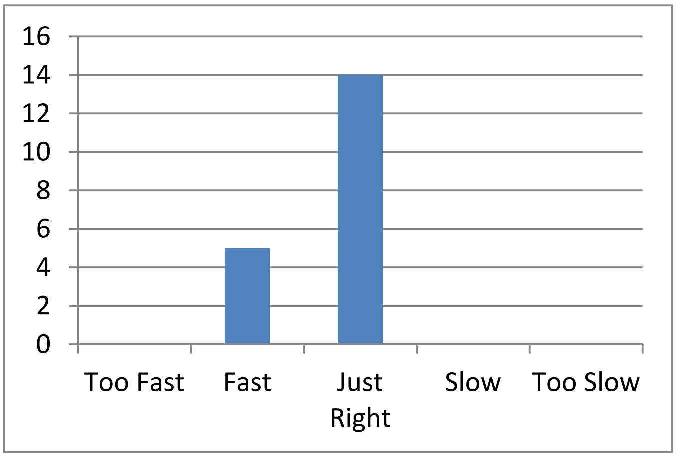
	x^2	$+2x$	$+1$
x	x^3	$+2x^2$	$+x$
+3	$+3x^2$	$+6x$	$+3$

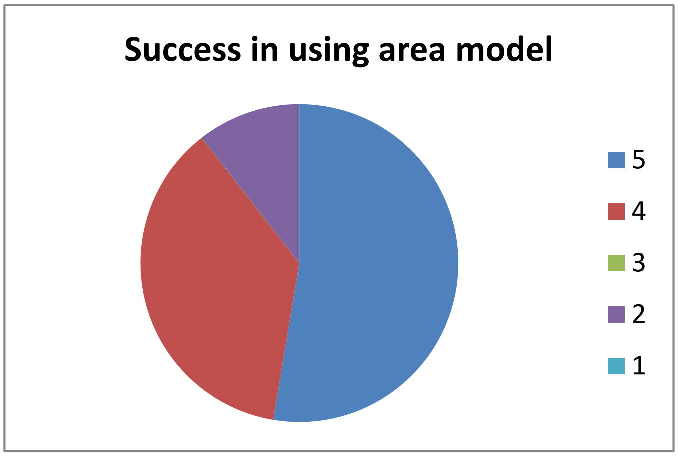
Note: Colour boxes are used in each array to help students recognise the location of similar terms. This is an important step in students being able to use the area model to multiply and divide.

12. Post-lesson reflection

General findings

- The lesson was a very positive experience for all students. In the student survey all students rated the lesson as 8+.
- The lesson was easy to follow for most students. Students understood what they were trying to do at all times. Students were asked to rate the pace of the lesson and to rate how successful they were in using the area model. Here is a summary of their answers:



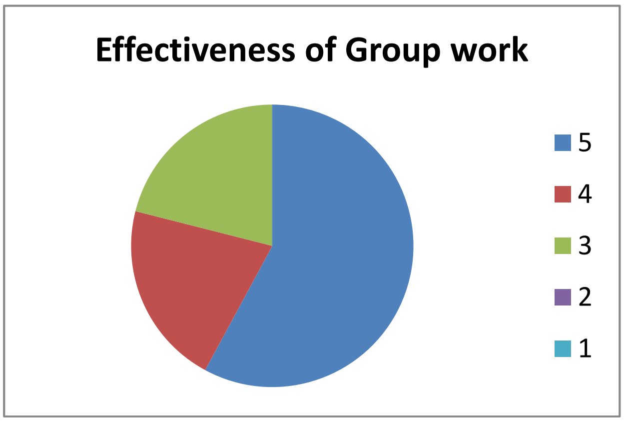


Students were fully engaged with the lesson content.

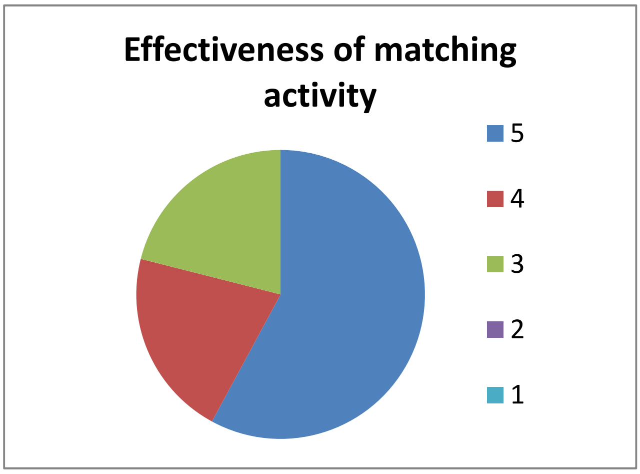
- Students followed instruction very well.
- The teacher-student interaction was excellent with all students willing to ask questions, answer questions and explain their reasoning.
- The use of colour to identify like terms in the array helped students understand the process of multiplication and division using an array. The use of loops by students to do same yielded similar benefits.
- Some students had difficulties filling in the arrays where understanding of integers was required. The progress of these students in the lesson was slowed by their lack of ability with integer number

operations.

- A small number of students had difficulties differentiating between multiplication and addition when filling in the array. This confusion tended to happen when students were splitting up the x term into two separate parts along the diagonal of the array.
- When checking their work (by examining the final term in the array) some students randomly changed the signs of certain terms so that the check worked. They failed to then check that the other terms were still correct given these changes. Some students abandoned basic number sense to "get" their check to work.
- Students worked well independently and then were happy to compare their work to that of their partner. In cases where answers did not agree most students proceeded to check their work and discuss what error had occurred. A small number of students simply changed their answer to match that of their partner when answers disagreed.
- Group worked proved very effective in most cases as it allowed students to discuss what they were learning.



- The matching activity proved a great success and was a great way to finish the lesson. Students worked well together and many were observed checking their partner's suggestions for which term should go where. As students worked on the matching activity they had to do some basic algebraic multiplication and addition in their heads which was a worthwhile exercise in itself. Students also had to think about the placement of different terms in the array which can only have helped solidify their learning. We had thought before the lesson that some student might not find the matching activity sufficiently challenging, however this turned out not to be so. All students worked up to the end of class on the activity and were fully engaged.
  Students were asked about the helpfulness of the matching activity [on a scale of 1 – 5] here are the results:



Recommendations

If we were to teach the lesson there are very few changes we would make:

- It might be a good idea to spend some time in the lead up to this lesson practicing basic integer operations, particularly for students who have experienced difficulties with this topic in the pass.
- The timing of the lesson was excellent with a lot of content fitted in. If possible it would be nice to give students a little more time in thinking about some of the important features of the area model. For example, giving students time to discuss why a cubic expression divided by a linear expression requires a 3×2 array would be good.
- This lesson was presented to a higher-level third-year group. Some students found the area model somewhat difficult due to lack of knowledge of integers. It would be interesting to see if an area model would be as effective with students of lesser ability, where integer problems are more likely.

Final conclusions

- The area model worked very well for division of algebraic expressions in a group of higher-level students.
- The area model requires students to have good knowledge of integer operations.
- Students get a lot of practice of basic algebra operations while practicing the area model.
- The effectiveness of the area model for students of lesser mathematical ability is untested in this case.

Worksheet Name:

Q. 1.
Divide $x^2 + 5x + 6$ *by* $(x + 2)$

x		
+2		

Q. 2.
Divide $x^2 + 7x + 12$ *by* $(x + 3)$

Q. 3.
Divide $x^2 + 8x + 12$ *by* $(x + 3)$

Note: Once finished do Extra Questions (a), (b) & (c) below

Extra Questions (in Copybook)

(a) $2x^2 + 7x + 3 \;\div\; (2x + 1)$

(b) $2x^2 + 9x + 9 \;\div\; (2x + 3)$

(c) $2x^2 + 11x + 12 \;\div\; (2x + 3)$

Q. 7.
Divide $x^2 + 2x - 15$ *by* $(x + 5)$

Q. 4.
Divide $x^2 - 3x - 10$ *by* $(x - 5)$

Q. 5.
Divide $x^2 - 7x + 12$ *by* $(x - 3)$

Q. 6.
Divide $x^2 - 9x + 20$ *by* $(x - 4)$

Extra Questions (in Copybook)

(d) $2x^2 - 7x + 5 \;\div\; (2x - 1)$

(e) $3x^2 + x - 4 \;\div\; (3x - 4)$

(f) $2x^2 - 3x - 20 \;\div\; (x - 4)$

Q. 11.
Divide $x^2 - 6x + 9$ *by* $(x - 3)$

Q. 8. **Divide $x^2 - x - 6$ by $(x - 3)$**	**Q. 12.** **Divide $x^2 - 11x + 30$ by $(x + 5)$**
Q. 9. **Divide $x^2 - 15x + 56$ by $(x - 9)$**	**Q. 13.** **Divide $x^2 - 7x + 10$ by $(x - 2)$**
Q. 10. **Divide $x^2 + 2x - 63$ by $(x + 7)$**	**Q. 14.** **Divide $x^2 - 10x + 24$ by $(x - 8)$**

Extra Questions (in copybook):

Q.15. $x^3 + 4x^2 + 3x - 20 \div (x + 1)$

Q.16. $x^3 + x^2 - 14x - 24 \div (x + 3)$

Q.17. $x^3 - 13x^2 + 56x - 80 \div (x - 4)$