Introducing quadratic functions through problem solving

Reflections on practice 2015

Lesson Plan for [Second Year, Quadratic Functions and Equations**]**

For the lesson on [18/03/2015]
At [Good Counsel College, New Ross, Co. Wexford], [Sean MacCormaic's] class
Teacher: [Sean MacCormaic]
Lesson plan developed by: [Sean MacCormaic, Brigid Kennedy, Shirley Dempsey, Sinead Furlong]

1. **Title of the Lesson: Introducing quadratic functions through problem solving**

2. **Brief description of the lesson:** To help students understand the relevance of quadratic functions to real life and the importance of the critical points of a quadratic graph.

3. **Aims of the Lesson:**

Short-term aims
I'd like my students to recognise quadratic functions.
I'd like my students to recognise the graph of a quadratic function.
I'd like my students to understand what the term "roots" means in relation to a quadratic function.
I'd like my students to understand what the term "roots" means in relation to the graph of a quadratic function.
I'd like my students to understand the importance of the roots of a function.
I'd like my students to be able to interpret the graph of a quadratic function.

Long-term aims
I'd like my students to see that maths is used in the real world.
I'd like my students to learn to be independent learners.
I'd like my students to learn to work together effectively.
I'd like my students to ask questions of the teacher and each other.
I'd like my students to learn to try different approaches to solving a problem.
I'd like my students to learn to explain their ideas.
I'd like my students to connect what they are learning to things they already know.

4. **Learning Outcomes:**

As a result of studying this topic students will be able to:

- understand that quadratic functions are used to describe real-world situations
- recognize a quadratic function from its algebra
- recognize the graph of a quadratic function
- interpret quadratic graphs in different contexts
- identify critical points of a quadratic graph
- understand the relevance and importance of critical points in context

5. **Background and Rationale**

Students first meet quadratic functions and equations in second year. Traditionally students are presented with a number of quadratic functions and are asked to factorise them. They are then presented with quadratic equations and are shown how factorisation allows you to "solve" these equations. Very often students are not presented with why quadratic functions and equations are important, where they are used or how to apply quadratic functions and equations to solve real-world problems.
We are going to try approaching the introduction of quadratic functions and equations differently. We plan to start with a simple contextualised problem, share with students the fact that this problem is modelled by a quadratic function and through a series of simple questions allow students to learn to

interpret the graph of the quadratic function and to identify the critical points of the graph and understand why these points are important.
We hope that by taking this approach students will have a better understanding of quadratic functions and find their graphs easier to interpret. This "for understanding" introduction should lay the groundwork for the formal algebra techniques associated with quadratic functions i.e. that of factorisation and solving or finding roots.

6. Research

The lesson is based around three simple in-context quadratic problems. The first problem is based around a short video which students are shown. This video is intended to engage students with the lesson material. The video used can be viewed at

http://video.nationalgeographic.com/video/weirdest-cape-gannets.

The first two problems are based on projectiles in flight. They are designed to be visual so that students can understand them more easily. The third problem is based on using a quadratic function to model profit and loss. This example is chosen deliberately as being more abstract to let students see that quadratic functions are used to model lots of different events.

We will use a small number of animated PowerPoint slides to support students' learning in the lesson.

7. About the Unit and the Lesson

This lesson aims to give students an intuitive understanding of quadratic graphs before moving onto the algebraic techniques used to solve problems relating to quadratic functions. It is hoped that by introducing quadratic functions and their graphs in context that students will gain a deeper understanding and also be able to bring some basic problem-solving to learning about quadratics. The lesson will focus on three in-context problems with accompanying graphs and students will be asked to answer a number of questions around each context. These questions will lead to the discovery of the critical points of a quadratic graph, including the roots, the turning point and the y-intercept.

8. Flow of the Unit:

Lesson		# of lesson periods
	Solving quadratic equations	
1	• The use of suitable strategies (graphic, numeric algebraic, mental) in the solution of quadratic equations of the form $x^2 + bx + c = 0$ where $b, c \in Z$ and $x^2 + bx + c$ is factorisable.	8 x 30min. The first of which is the research lesson
2	• Solve simple problems leading to quadratic equations	3 x 30min.

9. Flow of the Lesson

Teaching Activity	Points of Consideration
1. Introduction Students are shown an example of a quadratic expression on the board and asked if they have ever seen something like this before.	Students should identify the expression as quadratic or recognize that they have seen these expressions before when multiplying out a pair of brackets in algebra. Students should recognize the presence of a x^2 term as meaning it comes from a product.
Students are shown the graph of a quadratic function. The teacher explains that the shape of the graph is a parabola and that these shapes often occur in nature.	Students should identify the shape of the graph.
The teacher highlights two points on each graph as being the roots of the function and asks the students why these points might be important. The teacher explains that the values of x for which the function crosses the x-axis.	Some students might understand what the roots mean.
2. Posing Task 1 Students are shown a video of a gannet diving into water for fish. The trajectory of the gannet's flight is a parabola and so we can describe its flight using the maths of a quadratic function. We are going to describe the gannet's flight using the quadratic function: $h(t) = t^2 - 8t + 12$. Working in pairs, look at the graph of the gannet's flight and answer the following questions: (a) How many seconds passed before the gannet entered the water? (b) After how many seconds did the gannet come back out of the water? (c) How long was the gannet underwater for? (d) What was the deepest dive of the gannet? (e) After how many seconds was the gannet at its deepest? (f) After 7 seconds the gannet burped and dropped its fish. At what height did this occur?	Students should be fully engaged with the video. An image of the sea is superimposed on the graph of the quadratic function. The teacher explains that the x-axis represents time and the y-axis represents height. The teacher circulates the room to see if students are having difficulty answering the questions.
3. Anticipated Student Responses (a) 2 seconds (b) 6 seconds (c) 4 seconds (d) -4, 4 metres	As students give answers the teacher encourages them to explain their thinking.

(e) 4 seconds (f) 5 metres, approximately 5 metres, somewhere between 4 and 6 metres	
4. Comparing and Discussing Different students are asked for their answers to question (a) and question (b). The teacher marks in these answers on the graph on the whiteboard. The teacher asks the students why these points are important. The teacher reminds students that these points are called the roots of the quadratic function and emphasizes that the roots are a really important characteristic of any function. The teacher asks the students to describe what the roots mean, in the context of the story and also in general. Different students are asked for their answers to questions (d) and (e). The teacher identifies the turning point of the function and marks it in on the graph. The teacher asks the students why it's called the turning point. The teacher identifies the turning point as the minimum point of the function and asks students to explain why it's called the minimum point. The teacher asks students to explain what the turning point of the graph of a function means in general. The teacher asks students to explain what the minimum point of a function means in general. Different students are asked for their answers to questions (c) and (f). Students are asked to explain how they answered these questions. The teacher marks the solutions in on the graph on the whiteboard.	We want students to recognize the importance of the roots of a function and that they describe something really important about a function. We want students to explain what the roots mean in terms of the story. The image of the sea is removed from the graph to allow students to see the roots in general terms. We want students to explain what the roots mean in general –i.e. where the function changes from positive to negative or from negative to positive. The teacher superimposes the image of the sea on the graph again. We want students to be able to explain that the turning point is where the bird turns around. We want students to explain that the turning point is the minimum height of the bird (as opposed to the maximum depth). The teacher removes the image of the sea from the graph so that students can discuss the turning point in more general terms. We want students to understand that the turning point is where the function turns around or where it stops decreasing and starts increasing again. We want students to understand that the minimum point is the point where the function has its minimum value. We want students to be able to interpret the graph for themselves without having to be told how to do each question.
5. Posing Task 2 We are going to look at another example of a projectile flying through the air. This time the projectile is a Tomahawk missile instead of a bird. The missile's flight is also parabolic.	A PowerPoint slide of the missile's trajectory is shown on the board. We want students to understand that a parabolic

What type of function can be used to model the missile's flight? The quadratic function which describes the missile's height over time is $h(t) = -t^2 + 6t$ Here is the graph of the missile's trajectory. Working in pairs I want you to answer the following questions. (a) Why is it important to know the roots of this function? What do the roots tell us? (b) How long was the missile in flight for? (c) What was the maximum height reached by the missile? (d) After how many seconds did the missile reach its maximum height? (e) The early warning system alarmed when the missile was descending and reached a height of 8 km. How much time was there before impact?	graph is modeled by a quadratic function. The axes are labelled as time (x-axis) and height (y-axis). The teacher circulates around the room to check if students understand what they are doing. If some groups complete this task before others they may be given some extension questions. For example: Extension Question 1: Missiles only become detectable by RADAR above a height of 8 km. How many seconds of warning will a ship equipped with RADAR have before being struck by the missile? Extension Question 2: A ship can move at a speed of 150 ms^{-1}. To be safe from a missile strike, it must be at least 500m away from the impact location. Will the ship make it to safety in time?
6. Anticipated Student Responses (a) The roots tell us when the missile was launched and when it will hit its target. (b) 6 seconds. (c) 9 km. (d) 3 seconds. (e) 2 seconds, 4 seconds (incorrect).	As students give answers the teacher asks them to explain their reasoning.
7. Comparing and Discussing Different students are asked for their answers to question (a). The teacher marks in the roots on the graph on the whiteboard. The teacher asks the students to explain the meaning of the roots in this context. Different students are asked for their answers to question (b) and are encouraged to explain their reasoning. Different students are asked for their answers to question (c) and question (d). The teacher marks in these answers on the graph on the whiteboard. The teacher asks the students to give a mathematical name to this point. The teacher suggests that the turning point could also be given another name and asks students to suggest this name. Different students are asked for their answers to question (e).	We want students to be able to use the term "roots" and to be able to explain what the roots tell us about the context given and more generally. We want students to be comfortable explaining their reasoning. We want students to be able to use the term "turning point" and to be able to explain what the turning point tells us about the context and more generally. We want students to understand that this turning point could also be called a maximum point. Some students might mistakenly identify the earlier time at which a height of 8km occurs as a possible answer. It is important to discuss why this is not

Students are encouraged to explain how they came up with their answer.	correct in the context of the question. We want students to be comfortable explaining their reasoning.
8. Posing Task 3 We are now going to look at other places quadratic functions are used to describe the real world. Quadratic functions are often used in business plans to describe profit and loss. Some businesses only operate for a short amount of time each year. Could you think of any examples of such businesses? Different students are asked for examples of such seasonal businesses. The example we're going to look at is a business which manufactures and sells Irish flags, shamrocks, inflatable green hammers, leprechaun hats etc. The profit of the company over a number of weeks rises and then falls. Can you explain why this might be? Could you describe the shape of the graph? What type of function could be used to produce this graph? The profit of the company (in thousands of euro) is described by the quadratic function $p(x) = -x^2 + 6x - 5$ Working in pairs, I want you to answer the following questions: (a) What do the roots represent in this context? (b) Where does the graph intercept the y-axis? What do you think this value represents in the context? (c) Which week is most profitable? (d) What was the profit in this week? (e) If you were running the company when would you stop manufacturing flags, shamrocks etc.? Explain your reasoning. (f) In which week do you think St. Patrick's day occurs? Explain your reasoning.	This example is deliberately chosen to be more abstract to see if students can apply prior knowledge in a different context. We want students to understand that maths is applicable to real-world events. We want students to think of examples where the profitability of a business is short-lived. A slide showing different examples of short-lived businesses is displayed. The graph of the profit function is displayed on the whiteboard. The axes are labelled as time (x-axis) and profit (y-axis). We want students to relate the graph to the real world. We want students to recognize the graph as parabolic. We want students to relate a quadratic function to a parabolic graph. The teacher circulates the room to check that students understand what they are doing.
3. Anticipated Student Responses (a) When the company starts making a profit and stops making a profit. (b) -5, 5, -5000.	As students give answers ask them to explain their reasoning.

How much money they lose at the start. How much they spend on stock. The start-up costs of the business. They don't sell very much. (c) Week 3 (d) 4,000 (e) Week 5 – because I'll make a loss after that. Week 3 – because my profit is falling. Week 4 – because I want to sell any .remaining stock. (f) Week 3, week4, week 5	
4. Comparing and Discussing The teacher marks in the roots of the function and asks students to explain what the roots represent in terms of the question. The teacher marks in the y-intercept of the function and asks students what this value represents. The teacher marks in the turning point of the function and asks students to explain what the turning point represents in terms of the question. The teacher asks students when they would stop manufacturing stock. The teacher asks students when they think St. Patrick's Day occurs.	We want students to recognize the importance of the roots of a function and that they describe something really important about a function. We want students to describe the roots as where the company stops losing money and starts earning money and vice versa. We want students to understand that the y-intercept also tells us something important about the context. It tells us the starting value. We want students to recognize the importance of the turning point of the function and that it describes something really important about the function. We want students to describe the turning point as the point where the company is making the highest weekly profit. We want students to understand that their opinions are important and that different people have different opinions. We want students to be comfortable explaining their reasoning. We want students to be able to interpret the graph in the context of the problem. We want students to be comfortable thinking about an open-ended problem, offering a solution and explaining their reasoning.
5. Summing up What have we learned today? Quadratic functions are used in lots of places to describe real-life events. Can you give me some examples of where quadratic functions are used?	We want students to recognize the use of quadratic functions and their graphs to describe real-world events. We want students to recall that quadratic expressions come from multiplying a pair of brackets. We want students to recognize the algebra of a quadratic expression as the highest power of the variable being two.

The graphs of quadratic functions all have a similar shape. Can you describe this shape? Can you sketch me an example of a quadratic graph?	We want students to identify quadratics as being useful for describing projectile motion and profit-loss scenarios. We want students to understand that quadratic functions are parabolic in shape (i.e. u-shaped or n-shaped).
There are some points on the graph of a quadratic function which are very useful to know about. What are these points? What do the roots tell us about a quadratic function?	We want students to understand that the roots and turning point of a quadratic function are important points. We want students to understand that the roots are where the function crosses the x-axis or where the function changes from being negative to positive and back again.
What does the turning point tell us about a quadratic function? There are two different types of turning point. Can you tell me what these are?	We want students to understand that the turning point of a quadratic function is where the function stops increasing and starts decreasing or stops decreasing and starts increasing again. We want students to understand the terms "minimum point" and "maximum point".

10. Evaluation

There will be four observers present in the room not including the teacher who is teaching the lesson. Observer 1 will take notes on the flow of the lesson and on the student-teacher interaction. This will include observing the types of questions the teacher asks, the type of answers students give, if there is full engagement with the lesson by students, the type of questions students ask etc.
Observers 2, 3 and 4 will take different areas of the classroom and observe how students work together, what difficulties they might have, the level of discussion being had, if students are engaged with the lesson, if students understand what they are doing et.

11. Board Plan

See PowerPoint slides.

12. Post-lesson reflection

The lesson was deemed to be very successful but with some room for improvement.

Positive observations from the lesson

- Presenting quadratic functions and graphs in context can help students understand the features of quadratics much more easily
- In general most students were engaged with the lesson
- Excellent understanding of the concepts being taught
- Students exhibited little difficulty in interpreting graphs in the contexts that were presented

- Students did a lot of thinking for themselves
- Students recognised that maths is useful for describing real-world events
- In general students worked well in groups and discussed the problems they worked on
- Students offered different opinions on their maths and were willing to back up their opinions with solid reasoning

There was a high level of student engagement with the content material of the lesson. Students had never experienced quadratic expression before (except when performing the task of multiplying a pair of brackets) and yet by the end of the lesson most were comfortable interpreting the graphs of quadratic functions in context. The video clip used to set the scene for the lesson worked very well. Students were hooked by it and found it interesting. By using a real-life context students found it easier to then interpret characteristics of the quadratic function without a lot of support.

Most students had little difficulty in identifying the roots of the function in the context of the question. Most students had little difficulty in identifying the turning point of the graph in the context of the question.
At the end of the first task the importance of the roots and turning point was made and the associated vocabulary introduced. Most students seemed to recognise the importance of these critical points and were able to explain what these points represented in the context of the question and more generally.

When students were asked for the deepest dive of the gannet some students answered “4 metres” and others answered “-4 metres”. There was an interesting debate as to who was correct and two students commented that the negative sign tells you that the gannet is below the water line and that the answer should be “4 metres”

When students were given additional questions relating to the graph (e.g. how long was the bird under water for or at what height did the bird drop the fish) they had little difficulty in answering them. This strongly suggests that by letting students understand the function in its context allowed them to think for themselves and problem solve accordingly.

The majority of students had little difficulty with the second activity. In some cases extension questions were used to challenge the more able students. Students were quickly able to answer questions pertaining to the roots and the turning point of the function. The extension questions provided some students with the opportunity to extend their learning and to make connections with other areas of mathematics. The relevance of roots and turning points was reinforced from the first activity.

The third activity focused on using a quadratic function and graph to model the profit/loss of a company. This task was chosen purposely so that students could see that quadratic functions have a wide range of applications in describing real-world events. Most students performed really well in this task and it was evident that many of them had little difficulty in interpreting the graph of the function. Students understood that the parts of the graph beneath the x-axis represented a loss made by the company while the part of the graph above the x-axis represented the profit made by the company. Students intuitively understood the roots as the boundary between these two events. It was surprising that students had such little difficulty with this activity as the situation being modelled was more abstract than in the first two tasks.

Students naturally understood the turning point as the point at which maximum profit occurred.
The more open ended questions relating to when the company should stop manufacturing new product and when St. Patrick’s Day was likely to occur provided a nice opportunity for students to offer their opinion on the maths that they were learning. These questions also helped reinforce the idea that maths is used to describe real-life events.

Recommendations for improving the lesson

- Use of show-me boards to ensure all students are engaged with the lesson and contribute to the class discussion
- Increase wait time so that all students have time to consider the questions being asked
- Give each group of students their own worksheet with copies of each graph and encourage them to mark in the answers to each part of the activity on their graphs
- Changing the quadratic in the first activity to reflect the information presented in the video or explaining to students that our example is different to that presented in the video
- Giving students an additional activity where they are given some stories and related information and are asked to sketch the associated graph
- Follow the lesson up with suitable homework to reinforce the concepts studied during the lesson
- Follow the lesson up with an activity where the roots are determined using algebra. This should be based on recognising that quadratic expressions come from the product of two brackets and identifying, through inspection that writing a quadratic in factor form allows us to identify the roots

There are a number of ways in which we think the lesson could be improved. A very small minority of students weren't fully engaged with the lesson and we feel that more effective use of show-me boards could help to make students more responsible for their own learning. If we were to teach the lesson again we would emphasise to students at the start of the lesson that each group will be expected to contribute to the classroom discussion. We would tell students that there is no need for them to raise their hands when a question is asked, rather they will be given one minute to discuss the question with their partner and to write their answer on their show-me board. The teacher will then decide which group will be called on to provide answers. In this way, students need to maintain their focus on the lesson at all times and the teacher can use the show-me board to ensure a range of answers are presented, to identify students having difficulty and to highlight examples of exceptional thinking. The teacher-student interaction tended to be dominated by the same students all the time. By using show-me boards as a guide it would be possible to ensure the involvement of the entire class in both activities and feedback.

We think it would be a good idea to provide each pair of students with a copy of the graph of each function. We would encourage students to highlight where their answers exist on their graphs. By doing this we feel that students would better retain the information about roots and turning points.

On the first activity one student answered "60 foot" to the question "What was the maximum depth reached by the gannet?". Initially it was unclear where this answer came from but on reflection the student had remembered this fact from the introductory video. The quadratic used for the activity did not reflect the information presented in the video. It is reasonable that some students may have been confused by this. This problem is easily remedied, either by changing the quadratic used so that it reflects the information presented in the video or by explaining to students that our story is about a different dive / gannet.

We feel that we could have gotten students thinking about the application of quadratic functions to real life a little more. A simple activity where students are given a story with some basic information and asked to sketch the quadratic graph which represents this information could provide the opportunity for a lot of learning. Such an activity would form a nice close to the lesson.

When we started designing this lesson we weren't sure if the approach we used would benefit students. On the basis of this lesson we feel that it worked extremely well indeed. With this in mind we would recommend that a similar approach should be tried out when introducing other algebraic concepts. In Irish maths classrooms it has been normal to teach the theory & skills before the context. Often when students are asked to then apply what they know to a context they find it very difficult. Presenting quadratics in context first seems to provide students with a better understanding of what

they’re doing and why they’re doing it. It remains to be seen if doing so can help students understand, apply and retain their algebraic skills.