2^{nd} year Maths – Simultaneous Equations

For the lesson on 10/01/16 Tuesday at 14.25 – 15.05,
Blessington Community College, 2^{nd} year Mathematics
Teacher: Orla King
Lesson plan developed by: Catherine Byrne, Sarah Kavanagh and Orla King

1. **Coffee and tea-simultaneously**

2. **Brief description of the lesson:** Students will be presented with the problem below on a power point slide. They must apply their wide range of mathematical knowledge to solve it in a various number of ways. Pupils will share their findings and explain how they arrived at their solutions in detail, using their mathematical language.

Tea and coffee

Sam buys two coffees and a tea. The total cost is €3.20.

Jane buys tea and coffee from the same shop. He buys three coffees and two teas. The total cost is €5.20.

Using both Sam and Jane's totals can you find the cost of a tea and a coffee?

See how many different ways you can arrive at the same solution.

3. **Aims of the Lesson:**
   By the end of the lesson pupils should:
   - Be aware that there is always several ways to approach any problem.
   - Appreciate that mathematics is very important in our everyday life.
   - Develop their ability to think creatively about various methods of problem solving.
   - Understand that people may see problems differently and still arrive at the same end result.
   - Improve their ability to explain their methodology clearly, while using key mathematical language.
   - Make a connection between co-ordinate geometry, plotting line graphs and solving simultaneous equations algebraically.
   - Be encouraged and enthusiastic to teach their peers and others their method, in a simple manner.
   - Feel a sense of satisfaction from discovery learning and have a will to become better

independent learners.
- Develop and use their own generalising strategies and ideas and consider those of others.

4. Learning Outcomes:

Students should be able to:

- Understand keywords when discussing mathematical topics.
- Use tables, diagrams and graphs to represent and analyse linear relations.
- Present the information the information given, using graphs.
- Set up two equations to solve the problem, and let variables represent the unknown values.
- Find the value of missing variables, using simultaneous equations.
- Find the value of missing variables, using their knowledge of plotting linear graphs.
- Decide if two linear relations have a common value.
- Present their solutions, confidently amongst peers and teacher, explaining and justifying methods, inferences and reasoning.

Background and Rationale

As with algebra in general, Simultaneous equations cause confusion for students as to the meaning of the letters and the idea that they are variables which take on values dependent upon the context. It was intended in this lesson to elicit from the students their approach to solving simultaneous equations in a new context, and how much of what they had been taught on the topic would transfer to their solutions.

In the past, examiners' report that pupils need to be prepared in the following areas: [Junior Certificate Examination 2015 – Mathematics – Chief Examiner's Report]:

Mathematics is not a list of discrete rules and definitions to be learned but rather *a series of interconnected principles* that can be understood and then *applied in a wide variety of contexts*…..

Students should *practise different ways of solving problems* – building up their arsenal of techniques on familiar problems will help them *to tackle unfamiliar ones*. Students at Higher level *should pay particular attention to algebraic methods of solving problems* ……

Students should get into the habit of showing supporting work at all times. This will help them tackle more difficult problems, and will allow them to check back for mistakes in their work……

……… When using trial and improvement, students should develop methods for systematically improving the answer. For example, does an increase in the input lead to an increase in the output? If so, this may allow the problem to be solved more quickly…..

……questions were amenable to solution by methods that did not involve algebraic manipulation (such as trial and improvement), and many candidates were successful in using these methods. However, when these non-routine problems *required algebraic manipulation to solve them*, as they often did at higher level, *candidates tended to struggle*. [Mathematics – Chief Examiner's Report]

- Particularly with this unit and lesson, in the past, sometimes pupils find it difficult to grasp any *understanding of simultaneous equations* when they are just shown the elimination method. They are unable to make the link between functions, patterns, graphs and equations unless given some guidance about the connections in mathematical content. It is very important we address this issue when teaching all topics and encourage pupils to make links.

4. Research

The resources used by the planning team included a wide variety of materials and methodologies to make this abstract concept of finding two unknowns more concrete and to show there are different ways to do this.

By reminding the second-year students of their prior knowledge we wished to build up their confidence in their ability to solve the task. Real coin imagery was used in the PowerPoint to prompt recollection of their experiences of shopping and to improve their mathematical self-efficacy.

Similarly, images of mathematical tools previously used were shown, such as the use of the Bar Model[1] to illustrate the alternative ways to work out the problem. This is a visual representation of a number using unit cubes, lines (10 cubes), sheets (100 cubes) and boxes (1000 cubes). They are used in maths education in a pictorial approach to solving addition and subtraction problems.

ICT resources were provided by GeoGebra,, used to graph the two options and the solution.

The topic is embedded in the syllabus in many aspects.

1. It relates to algebra and real life situations
2. It highlights the importance of finding many approaches to solving problems
3. It connects different solutions from different aspects of maths, graphs, numbers, mental arithmetic and algebra
4. It gives an opportunity to use graphs of functions and interpret equations
5. It familiarises the students with the terms and formats of equations.

These initiatives facilitate learning by connecting with the learners' prior learning, both academic and practical, which increased their self-belief and their mathematical self-efficacy.
It used many methods to appeal to different learning styles, such as active and visual learners, who may have found an abstract approach a challenge. It also offered a challenge to the students who learn in an abstract manner, as it offered another method to solve by substitution.

5. About the Unit and the Lesson

This unit was designed to emphasise the link between several strands in the JC syllabus, mainly focusing on geometry and algebra, while introducing pupils to solving simultaneous equations using real-life problems.

Geometry: Prior knowledge

p.20 J.C Syllabus 2.5 Synthesis and problem solving skills
− explore patterns and formulate conjectures
− explain findings
− justify conclusions
− communicate mathematics verbally and in written form
− apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
− analyse information presented verbally and translate it into mathematical form
− devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

p. 27 Algebra J.C Syllabus (Most Relevant sections only are given)

Topic	**Description of topic Students learn about**	**Learning outcomes Students should be able to**
4.1	Relations derived from some kind of context – familiar, everyday situations.	− use tables, diagrams and graphs as tools for representing and analysing linear,
4.2 Representing situations with tables, diagrams and graphs		− develop and use their own generalising strategies and ideas and consider those of others − present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.3 Finding formulae	Ways to express a general relationship arising from a pattern or context.	− find the underlying formula written in words from which the data are derived (linear relations)
4.4 Examining algebraic relationships	Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships.	− show that relations have features that can be represented in a variety of ways − distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulas expressed in words, and algebraically − use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others

6. Flow of the Unit: Where the lesson fits into the teaching plans

Lesson		# of lesson periods
1	• Co-ordinate Geometry (in depth understanding of y=mx + c, and when to apply this knowledge to solve problems) • Plotting 2 linear graphs, using y=mx +c, on a plane and analysing if they have a point of intersection.	3 x 30 min.

2	• Discussing feedback from Christmas test; covering topics on co-ordinate geometry solving linear equations.	1 x 30 min.
3	• Investigating different methods for solving simultaneous equations, by introducing the topic as a problem.	1 x 40 min. **(research lesson)**
4	• Discuss and analyse methods of solving simultaneous equations algebraically, using substitution and elimination methods. • Setting up expressions, and equations to solve written problems algebraically.	3 x 30 min.
5	• Present students with sunflower problem from algebra through the lens of functions 1, p.22.	1 x 30 min.

7. Flow of the Lesson 40 minute class

Teaching Activity	Points of Consideration
1. Introduction (5 mins) Prior knowledge is gathered from the class and written up on the board (5 mins)	Students may need help here to remember some essential parts of previous learning on simultaneous equations.
2. Posing the Task: (10 mins) The power point slide with problem (see beginning of this lesson proposal) is presented to the class. They are encouraged to take their time and show clearly the different ways they use. The teacher indicates that students can avail of any of the materials on the provisions table.	Students will recap on various methodologies they can use to approach problem solving questions. Materials are left on a table-cubes, rulers, graph paper for the students to use if they desire.
3. Anticipated Student Responses The flow of expected solutions in shown in the poster below, which was composed by the teachers involved. Allowances are made for unforeseen methods chosen by students and this makes the lesson more interesting. It will also be observed how well students work with the unknowns-how they represent "price of a coffee", for example and how they use these in equations or in a graph,	INCORRECT ANSWERS: Encourage pupils to be ready to explain their thought process with the teacher and class if chosen to do so; 'How did you get here?' 'Why did you decide to do______?' Acknowledge every attempt and how the workings out are shown, and problem solving strategy chosen. Ask pupils who have an incorrect answer to now try another method and don't comment on the previous solution. HINT: If pupils get stuck or require some food for thought, the teacher can pose the question, 'Can you think of some possible prices for the cost of a coffee and the cost of a tea?' This gives an entry point. They can also be shown the project maths problem solving poster again to give them some ideas.
4. Comparing and Discussing (Ceardaiucht) (20 mins) The teacher will choose different pupils who have a good, clear layout of their workings of one of the methods, or a new one, to come up to the board and run through their approach with their peers. When they are discussing their method in front of the class the teacher will encourage their use of various mathematical concepts and language .	EXTENSION QUESTIONS: Which of your solutions would you choose for a primary school class? Discuss which solution you think is most efficient. Discuss what key area in mathematics most pupils have gone with. Highlight use of keywords in explanation and good use of workings out to encourage other students to continue to do this in the future.

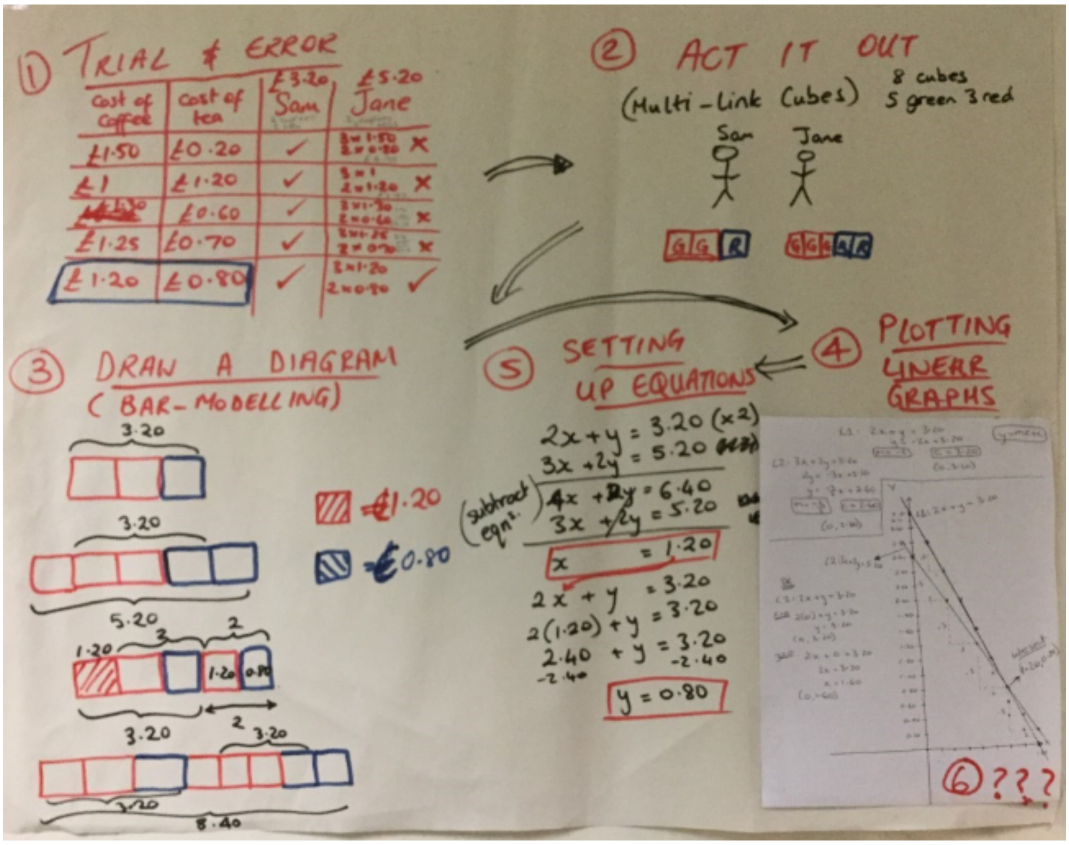
	The students will be staged in order of their strategy chosen and the higher order thinking needed to carry out each method.
5. Summing up (5 mins); The teacher encourages the students to take down any methods they did not get. Pupils will be asked to grade, using their fingers, how confident they would feel using the different approaches that their peers used, if given a similar problem, in the future. Homework is set.(See end of this proposal).	All the work is left on the board so that each method can be contrasted and compared with the others. Students are directed to come up with a sentence or two that states a difference/similarity between each approach.

8. **Evaluation**

- **What is your plan for observing students?**
  The two observing teachers will partition the class and, by use of a seating plan, take notes on what different students are doing- how they are tackling the task of translating from the words to mathematical symbols and what methods they choose first etc.

- **What additional kinds of evidence will be collected (e.g., student work and performance related to the learning goal)?**
  During the whole class discussion, and the presentations of students' methods by students chosen to do so by the teacher, the observers will be noting the use of mathematical language. A key concern will be how well students grasp the use of variables for the cost of tea and coffee.

9. Board Plan



9. Post-lesson reflection

- **What are the major patterns and tendencies in the evidence? Discuss**

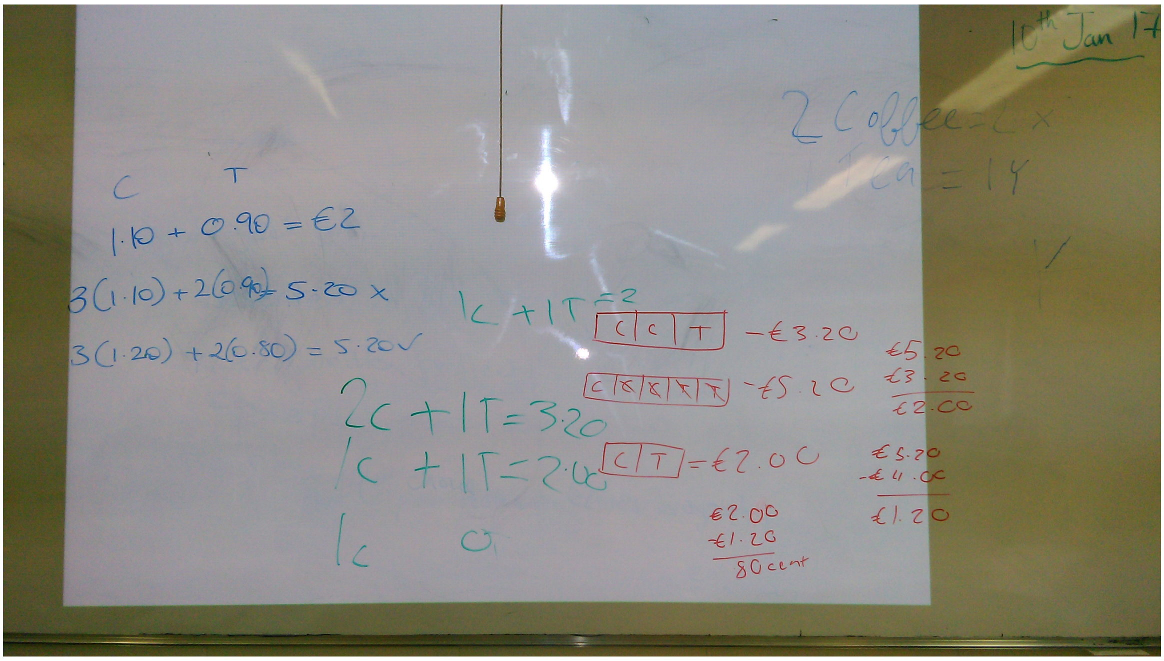
The students were grouped by mixed ability so there was a variety of responses within each group. This worked very well as there was a lively response to the problem from all groups as they all sought ways to solve the problem. The collaboration and sharing within the class and the attention they gave the teacher as she described the task showed that this activity was building on existing good practice of collaborative problem solving of real life problems.

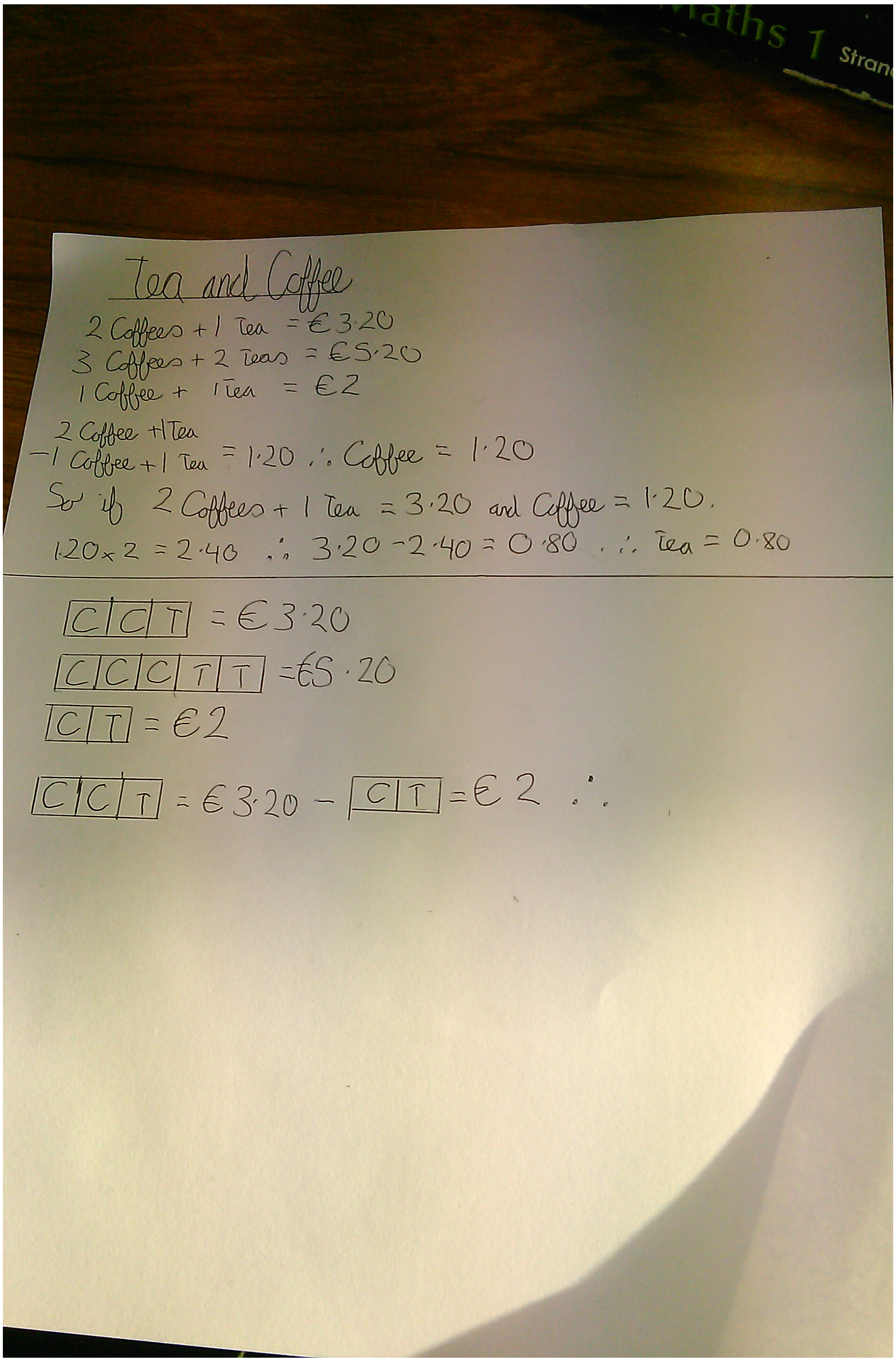
Unfortunately, the key patterns mentioned in the chief examiner's report are still evident here. Majority of the pupils could solve the problem using the trial and error method, which is commonly seen amongst ordinary and foundation level. However, only a few tried to access the problem formulating algebraic expressions and equations. I think this was because of the unit planning. If this lesson was placed at the end of the algebra unit pupils would have a greater knowledge of setting up equations and typically using x or y as the variables in a question.

While some pupils attempted to set up equations, there was a common misconception and tendency for pupils to label the variables as c and t. This illustrated that they linked the letter or unknown to the word they related it to and not the fact that the letter represented the cost of the drinks, and as one variable changed so did the other. This choice of unknown also meant it was harder for pupils to connect this problem with linear graphs.

Many students adopted the symbols T and c for tea and coffee, which has the potential risk of causing confusion as to their meaning-do they stand for drink types? Most students seemed to translate them into values in their solutions but it would be worth teasing out what they see the variables as exactly.
Also, some students later seemed to believe that c and T could only take on singular values, as with the consonants themselves. This is a step above seeing them as standing for objects but work would be needed to distinguish between letters standing for an unknown value and for a value that can vary.

Another example is given below of the use of the first letter of tea and coffee for the unknown values. This student has used the bar method to good effect.


10th Jan 17
C T
1.10 + 0.90 = €2
3(1.10) + 2(0.90) = 5.20 x
3(1.20) + 2(0.80) = 5.20 ✓
1C + 1T = 2
C C T = €3.20
C C C T T = €5.20
C T = €2.00
2C + 1T = 3.20
1C + 1T = 2.00
€5.20
€3.20
€2.00
€5.20
-€4.00
€1.20
€2.00
-€1.20
80 cent



Tea and Coffee

2 Coffees + 1 Tea = €3·20
3 Coffees + 2 Teas = €5·20
1 Coffee + 1 Tea = €2

2 Coffee + 1 Tea
−1 Coffee + 1 Tea = 1·20 ∴ Coffee = 1·20

So if 2 Coffees + 1 Tea = 3·20 and Coffee = 1·20.
1.20 × 2 = 2·40 ∴ 3·20 − 2·40 = 0·80 ∴ Tea = 0·80

C | C | T = €3·20
C | C | C | T | T = €5·20
C | T = €2

C | C | T = €3·20 − C | T = €2 ∴

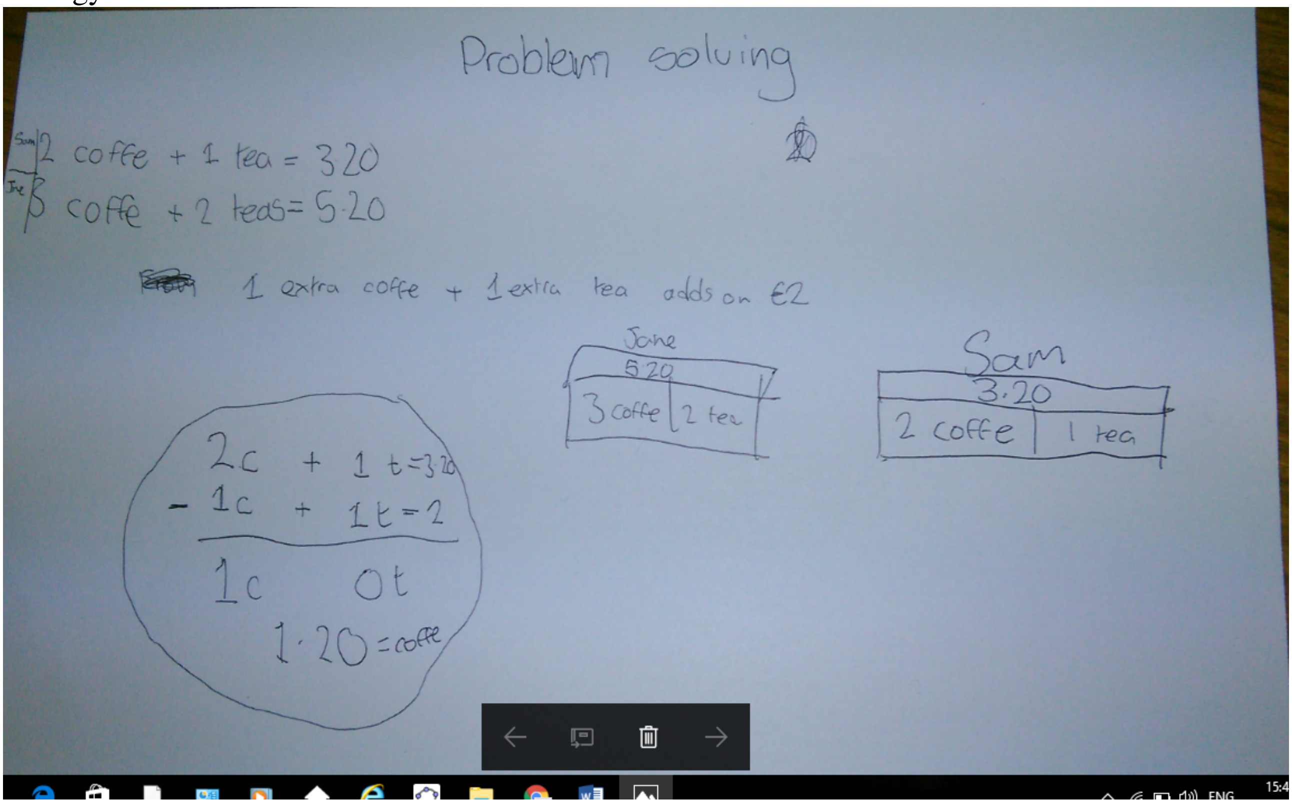
Another example (above) shows the use of ‘coffee’ and ‘tea’ instead of ‘cost of coffee’ etc. One solution is given as ‘Tea = 0.80’ instead of ‘Cost of a tea’. This does not necessarily imply that the student is not sure of what is being varied in the question but it begs for further research to see how other variables, especially in abstract equations, are interpreted.

Use of the letters c and t or the related names was the prevalent approach in the class when trial and error was not used. In fact no one used other letters, even the only two students who used a graph did not use x and y but stayed with c and t. It makes the understanding in this context easier, of course, but as long as the right significance is given to the symbols.

One step we intend to follow up with, and check their understanding of variables, is asking the students to verify their answers.

We are inspired to revise our lesson proposal and do more in-depth research the next time round. Being a researcher while teaching makes real sense and allows us to always have a part of ourselves looking in and questioning what we are doing, while another part is getting on with 'tried and tested' approaches, to the point that we have a willingness to forego those approaches in light of evidence, if needs be.

The reasoning in this question is made simpler by the values as shown in these solutions. As the chief examiner's report referred to early might say-the student (below) made systematic improvement on trial and review by seeing that "I extra coffee and 1 extra tea adds on €2" More challenging problems would follow-perhaps something like in the homework-where this particular strategy would not be available.



- **What does the evidence suggest about student thinking such as their misconceptions, difficulties, confusion, insights, surprising ideas, etc.?**

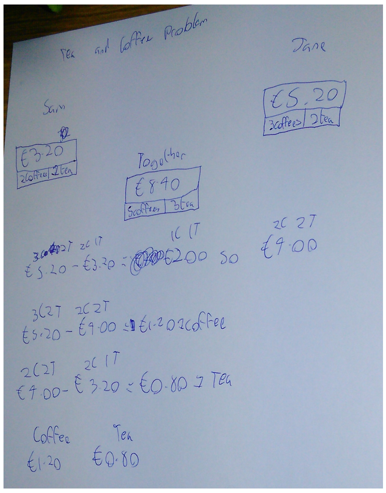
The evidence showed that students were thinking and adapting their strategies throughout the class, by their discussion and their ability to think outside the box. For example, students were opting to draw the problem out, to use the bar method, to scratch out one method and try a new method yet still were happy to share their early attempts with the observers.

- **In what ways did students achieve or not achieve the learning goals?**

Students did achieve their learning goals. At the end of the lesson, students were aware that there were several methods that could be applied to derive the correct answer and find the price of the tea and coffee per cup.

The students solved the problem as it was given and demonstrated that they had understood the challenge as it was given to them, as they got straight into it and tackled the challenge with enthusiasm. The real-life nature of the task, pricing tea and coffee, meant that it tapped into their experiences so they felt they could do it.

This allowed them to eliminate time wasting and arrive at the correct solution quicker. As with the pupil metioned above, the pupil (opposite) was also able to spot if the cost of one coffee and one tea must have been €2, the cost of two coffees and two teas must have been €4, allowing them to spot straight away then that one coffee clearly cost €1.20 when you take it away from €5.20.



Unfortunately, pupils were unable to link their knowledge of y = mx + c and line graphs to this problem since they chose c and t instead of x and y as the variables. I has anticipated that they might have come up with this solution themselves, so it allowed me to reflect on my teaching of co-ordinate geometry and focus more on how I could have introduced them to more problems involving real-life graphs now.

- **Based on your analysis, how would you change or revise the lesson?**

After careful reflection, there may be some changes to the lesson in future, although the lesson was very successful. We would consider projecting onto the screen to stimulate discussion in the class, maybe images related to the task or GeoGebra graphs of the problem. Overall the class was fully engaged and were very open to the new learning.

In the future, I think a class with greater prior knowledge in algebra and geometry would access this lesson with greater effect, so I fully intend to do a similar lesson once again after the midterm when my 2nd years have a greater knowledge of the algebraic content on the JC syllabus. The more pupils are in contact with such problems, in both familiar and unfamiliar contexts, the more they will be able to access it. I would also try to add another task in to the lesson that was not as such feasible to solve using trial and error to help facilitate pupils to try new means, since they are all quite confident with this method.

- **What are the implications for teaching in your field?**

Try as many innovations as possible and to build a level of trust and respect in the class and develop greater interest and curiosity in maths.

This lesson has highlighted that in a higher level 2nd year class, pupils are still unsure of how geometry and algebra can be linked. It is, therefore, very important that we as teachers try to rectify this matter and allow more time and set greater problems to *embed this early on so that we can begin to minimise the gap in pupils learning from forming between junior cert and leaving cert. Since pupils find it difficult to use algebraic reasoning at even junior cert higher level it is no wonder that this is an even greater issue for ordinary leaving cert students.*

When introducing pupils to variables in first year we must emphasise that they are just letters given to represent an unknown value in a certain problem and get rid of the misconception that when simplifying like terms 'we add a with a's; and b with b's; apples with apples, etc.'

We are inspired to revise our lesson proposal and do more in-depth research the next time round. Being a researcher while teaching makes real sense and allows us to always have a part of ourselves looking in and questioning what we are doing, while another part is getting on with 'tried and tested' approaches, to the point that we have a willingness to forego those approaches in light of evidence, if needs be.

POWER POINT SLIDES SHOWN ALONG SIDE METHODS DEMONSTRATED BY STUDENTS DURING THE LESSON:

Tea and coffee problem!

Sam buys two coffees and a tea. The total cost is €3.20.

Jane buys tea and coffee from the same shop. He buys three coffees and two teas. The total cost is €5.20.

Using both Sam and Jane's totals can you find the cost of a tea and a coffee?

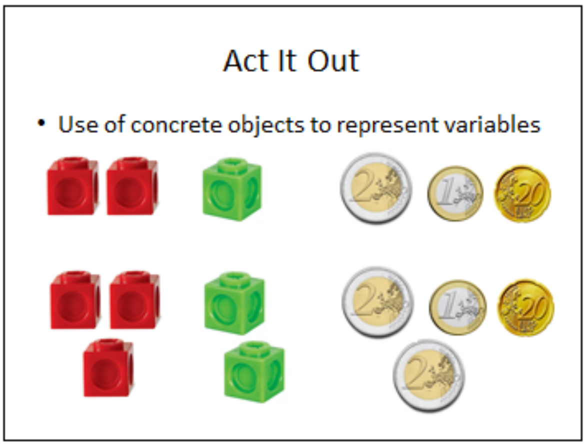
See how many different ways you can arrive at the same solution.

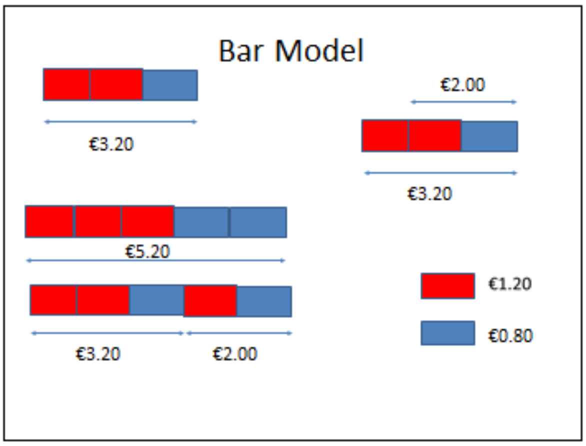
Extension

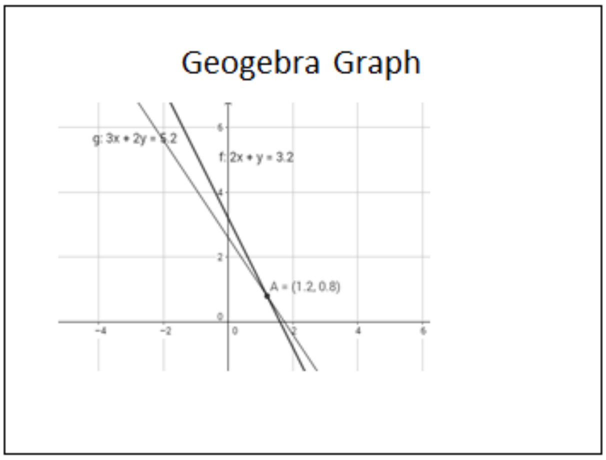
- Which of your solutions would you choose for a primary school class?
- Discuss which solution do you think is most efficient?

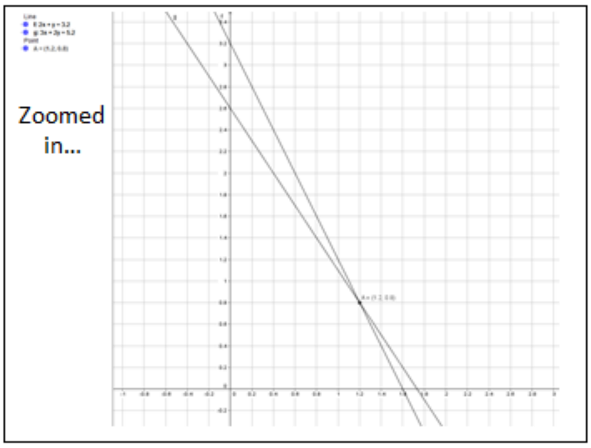
Trial and Error

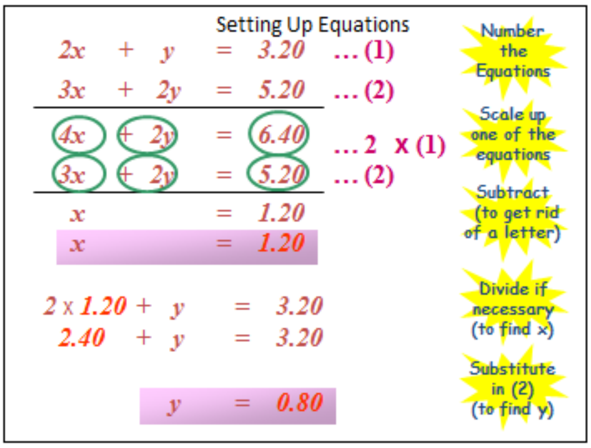
Cost of Coffee	Cost of Tea	€3.20 Sam (2 coffee & 1 tea)	€5.20 Jane (3 coffee & 2 tea)	
€1.50	€0.20	✓	3 x €1.50 2 x €0.20	4.90 X
€1	€1.20	✓	3 x €1 2 x €1.20	5.40 X
€1.30	€0.60	✓	3 x €1.30 2 x €0.70	5.30 X
€1.25	€0.70	✓	3 x €1.25 2 x €0.70	5.15 X
€1.20	€0.80	✓	3 x €1.20 2 x €0.80	5.20 ✓











HOMEWORK QUESTION::

Number puzzles

The larger of two numbers is twice the smaller, and the sum of the numbers IS 270. What is the smaller number?