Introduction to Differential Calculus
6th Year Ordinary Level

For the lesson on 6th of December 2016
At ScoilRuain, Killenaule
Teacher: Adrienne Cunningham
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1. **Title of the Lesson:** Introducing rates of change in non-linear functions.

2. **Brief description of the lesson:** How to use secants to best approximate the rate of change of a non-linear function plotted on a graph, followed by students discovering that the slope of a tangent at the point of interest gives the exact rate of change at that point.

3. Aims of the Lesson:

We would like our students to appreciate that mathematics can be used to solve real world problems.
We would like our students to experience meaningful mathematics i.e. that they see a need for what they are studying.
We would like our students to connect and review the concept of rate of change of a linear function that they have studied already.

We want our students to understand that the way to get the rate of change of a non-linear function at a point is to get the slope of the tangent at that point.

4. Learning Outcomes:

As a result of studying this topic students will recognise that:

a) The rate of change of a non-linear function changes
b) The rate of change can be approximated using secants to the curve
c) The exact rate of change at a point is calculated by finding the slope of the tangent to the curve at that point.

5. Background and Rationale

We chose this topic because students often struggle linking coordinate geometry and slope with rate of change when it comes to real life scenarios, even though they will have covered this at junior certificate level.
We also believe that (even though differentiation from first principles is no longer on the ordinary level syllabus) if a student can understand the link between the slope of a (tangent)line in co-ordinate geometry and the rate of change in calculus, then it will put them in a much stronger position to tackle further calculus. We feel students often see calculus as a somewhat abstract area of maths, however following this lesson, we hope more students will see the practical nature of the subject.

6. Research

Textbooks, the leaving certificate and junior certificate syllabi, the maths development team website (www.projectmaths.ie) and prior teaching knowledge and experience were all used in the formation of the lesson proposal. The maths development team document "Introduction to Calculus" was a major resource we looked at:
http://www.projectmaths.ie/documents/T&L/DifferentialCalculus.pdf

The maths development team document "Understanding Graphs" was a resource used for the prior knowledge.
http://www.projectmaths.ie/workshops/workshop4/UnderstandingGraphs.pdf

7. About the Unit and the Lesson

This research lesson is designed as a type one problem i.e. a productive problem that leads to a new idea for students by reflecting on its solutions; the new idea in this case being differential calculus.
The lesson asks the students to find the slope of different secants of the curve. This is designed to highlight the fact that the slope of the curve changes at different points. The aim is that students will discover that they can closely estimate the slope of the curve at a certain point by getting the slope of a very short secant which includes the point in question.
They will then discover that the slope of a tangent at that point gives the rate of change at that point.

After the research lesson approximately ten lessons will be used to cover calculus as specified on page 43 of the syllabus.

5.2 Calculus	– find first and second derivatives of linear, quadratic and cubic functions by rule – associate derivatives with slopes and tangent lines – apply differentiation to • rates of change • maxima and minima • curve sketching

In advance of the research lesson there will be 4 lessons given to investigating linear functions.

8. Flow of the Unit:

Lesson		# of lesson periods
1	• Revise graphs and functions/y intercept	2 x 40 min.
2	• Revise methods of getting the slope of a line	1 x 40 min.
3	• Revise speed, distance and time and quadratic expressions. Quick look at what a secant is and average speed. Give homework to lead into research lesson class.	1 x 40 min.
4	• Investigating rates of change in relation to non-linear functions	1 x 40 min. **(research lesson)**
5	• Discovery of the rule for differentiating polynomials and development of calculus	10 x 40 min.

9. Flow of the Lesson

Teaching Activity	Points of Consideration
1. Introduction Teacher will remind the students that in the previous three classes they revisited graphs, functions, distance, speed and time as well as co-ordinate geometry. Teacher will explain to students that today they will look at rates of change relating to functions which are **not** straight lines (non-linear). For homework students were asked to find coordinates on the curve t^3-t^2in the domain 0>t>8 and fill this into a table provided. This will be corrected in class.	
2. Posing the Task Students will work in pairs and be given a cubic function, table filled out from 0 to 8 and 6 images of the graph of the function. The first 2 graphs will have secants drawn on them to give some guidance to the students. Students will be asked to find 1. the average speed of the car from 0 to 8 seconds. 2. the average speed of the car from 0 to 4 s 3. the average speed of the car from 4 to 8 s 4. the average speed of the car from 1 to 7 s 5. the average speed of the car from 2 to 6 s 6. the average speed of the car from 3 to 5 s 7. their best estimate of the actual speed at 4 s	Ask students do they understand what they are being asked to do before they begin the task. Ask students do they know how to answer first question.

3. Anticipated Student Responses	
1. the average speed of the car from 0 to 8 seconds. R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 2. the average speed of the car from 0 to 4 seconds R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 3. the average speed of the car from 4 to 8 seconds R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 4. the average speed of the car from 1 to 7 seconds R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 5. the average speed of the car from 2 to 6 seconds R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 6. the average speed of the car from 3 to 5 seconds R1. Slope formula R2. Rise over Run R3. Distance divided by Time from table 7. we want to find out the actual speed at 4 seconds. What do you think is the best way to do this. R1. use answer from question 1 R2. An estimate based on questions 1 to 4 R3. Add results from questions 2 and 3 and divide by 2 R4. Using mid point of 3 and 5 to get 4 R5. Use answer from 3 to 5 R6. Using different intervals closer to 4 e.g. 3.5 to 4.5 R7. Gets the slope of the tangent drawn at 4	Teacher will monitor each pair's work with the seating plan looking for the anticipated responses. Students who get stuck may be slightly prompted.

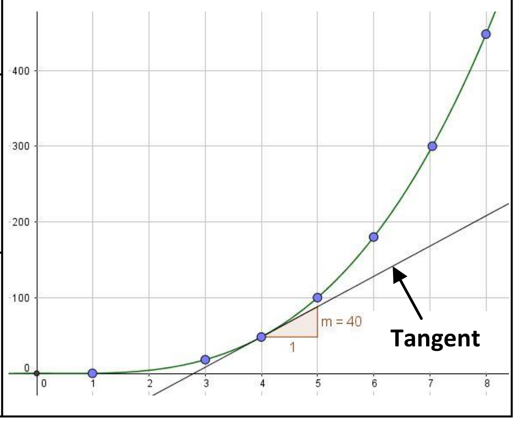
4. Comparing and Discussing Student solutions will be shared in the following order: **Questions 1-6**: Distance ÷ Time from table Slope formula Rise over Run **Question 7:** Use answer from question 1 An estimate based on questions 1 to 4 Add results from questions 2 and divide by 2 Use answer from 3 to 5 Using different decimal values closer to 4 Gets the slope of the tangent drawn at 4	 What is this value really telling us? How accurate is this? Does this result look familiar? (Same as 2 previous) Why is this a better estimate? What is this value really telling us? Why is this a better estimate? How close to 4 can we get? How can I draw a tangent by eye? We're getting the limit as the interval → zero.
5. Summing up Teacher will stress that the smaller the secant used to estimate the slope of the curve, the more accurate that estimate will be. Teacher will inform the students that there is a method for finding the exact slope of the curve at a given point (without the need for a table or graph) and they will be learning how in the next classes. Homework: Write down two new things you learned today.	

10. Evaluation

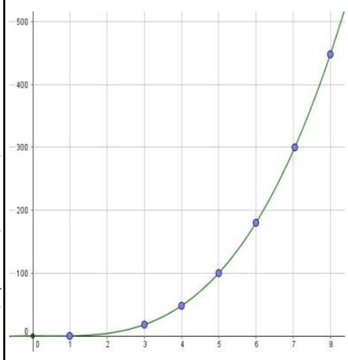
- There will be five observers in the room. One observer will use lesson note and may move around the room. The other observers will not move during the lesson.
- The stationary observers will observe 6 students each.
- At the end of the lesson, students will leave their work in the room so observers can have a look in more detail and keep a record of interesting work done.

11. Board Plan

Whiteboard

First 6 Qs	**How best to estimate speed at 4 seconds**	
Student 1 (D/T) 0 – 8: 448/8 = 56m/s	**Student 7** **Avg. for 8s** 0 – 8: 448/8 = 56m/s	**Student 11** **slope of the tangent to the curve when t = 4** **40 m/s !!**
Student 2 (Rise/Run) 0 – 4: 48/8 = 12m/s	**Student 8** **Mean of 0-4 & 4-8** (12+100)/2 = 56m/s	
Student 3 (Slope) 4–8: (448-48)/4 = 100m/s		
Student 4 1–7: 294/6 = 49m/s	**Student 9** **Avg. for 3-5s** (100-18)/2 = 41m/s	
Student 5 2–6: (180-4)/4 = 44m/s	**Student 10** **Avg. for e.g. 3.5-4.5s** (70.875-30.625)/1 = 40.25m/s	
Student 6 4–8: (448-48)/4 = 100m/s		

Green graphboard with graph plotted



12. Post-lesson reflection

<u>What are the major patterns and tendencies in the evidence? Discuss</u>

In order to conduct a successful lesson, the objective must be clearly outlined to the class. Initially, it was clear to all that we would look closely at the rate of change of non-linear functions. However, upon reflection some students were unsure of the ultimate goal of the excercise: to find the speed at 4s. This needs to be really highlighted in advance.

Some students were slow to begin their task – maybe because of observers in the room or a different teaching method. They relied a lot on initial steps their partners were making. Once they were assured they were making progress in the correct direction, they worked well. However, they were poor to work on the calculations in pairs. Most students worked alone and checked with their partners merely for reassurance.

We found that students mainly used the slope formula with a few using “rise over run”. We found that students approached drawing secants without any trouble.

The teacher could predict which students would make faster progress. Students were very engaged and on task.

What are the key observations or representative examples of student learning and thinking?
Students worked very well once the initial question was done. They realised they were using a correct method so their confidence received a boost which in turn got them working well through q 2-6.
Some students were very good to use the correct units of speed with each answer, however some omitted them. This was addressed by the teacher when needed.
The fact that speed varied over time was clear to all and was also highlighted by the teacher. Many students were unsure when it came to the last question - 'find the actual speed ***at*** 4 seconds'. This was worked through at the board by the teacher. As students did not utilise pair work too well, a lot of the class were stuck for time and just trying to approximate get something written down for the last question. Upon reflection, as a group, we feel we may have put too many questions in the task and therefore adequate time was not left for finding the speed ***at*** 4 seconds.

What does the evidence suggest about student thinking such as their misconceptions, difficulties, confusion, insights, surprising ideas etc.?
Students understood q2-6 well; problems with q1 & q7 were discussed above. Students also were confident using slope formula & rise over run method.

As mentioned above, misunderstandings occurred to do with the phrasing of q1 (because of 'average speed') and q7 (because of 'speed ***at*** 4 seconds') and these were dealt with effectively during the lesson and in subsequent lessons thereafter. Students had some difficulties getting used to a new style of teaching and observers in, for this lesson.

Unfortunately about one third of the class was absent for this research lesson (as the flu was rampant in the school at the time). Hence, some of the weaker students who would normally try to work quietly on their own were faced with pairing up with people they weren't used to working with.

In what ways did students achieve or not achieve the learning goals?
It was felt that the first goal a) below was completely met, goal b) was well met but goal c) was not really met. However, it was felt that this could be addressed in the following lesson. Some students were approaching goal c) by saying 41m/s is the best estimate of the speed at 4s.

Learning outcomes from lesson proposal document:
As a result of studying this topic students will recognise that:

a) The rate of change of a non-linear function changes

Students were aware of and understood that the rate of change changes by the end of this research lesson.

b) The rate of change can be approximated using secants to the curve

Students were very comfortable using secants to approximate the rate of change, even though this method was relatively new to them.

c) The exact rate of change at a point is calculated by finding the slope of the tangent to the curve at that point.

Most students were not able to find the slope of the tangent at that point by the end of the research lesson. However, with further pair work and show and tell exercises with the whole class present in the following days, all students in the class are now comfortable using this method to find the exact rate of change at a point.

Observers felt the flow worked well and the board work was very well presented and easy to follow (see appendices B & C). Lesson note confirmed that the timing of the first 40 minutes of the lesson went to plan. This helped students achieve the first two learning outcomes.

Based on your analysis, how would you change or revise the lesson?

- Possibly save some time by either the teacher doing the 0-4s and 4-8s secants on the board at the start or just give the rates of change for these time periods to the students.
- Attempt to make it less abstract by reminding them what 10m/s is (100m sprinter) or that 100km/hr means 28m/s.
  Maybe pictures of a race car speeding up across the bottom of the graph would help too.
- A table that could be filled in as the boardwork progresses with the slopes of the various secants could be a good point of reference.
- Extending secants either side of the curve might help with the transition towards a tangent at a point.
- Present pairs of students with one sheet and tell them to share the calculation work.
- More prior knowledge around the term 'average speed'
- At a certain point in the lesson tell them they must answer question 7 before the time is up and give a reason for their answer.
  "We want to find out the actual speed at 4 seconds. What do you think is the best way to do this?"
- One option would be to leave the tangent until the following lesson.
- For a higher level group we would use intervals of 4-8s, 4-7s, 4-6s, 4-5s etc. (to be consistent with first principles)

Students and teachers are new to this style of lesson and we all need more practice in its implementation.

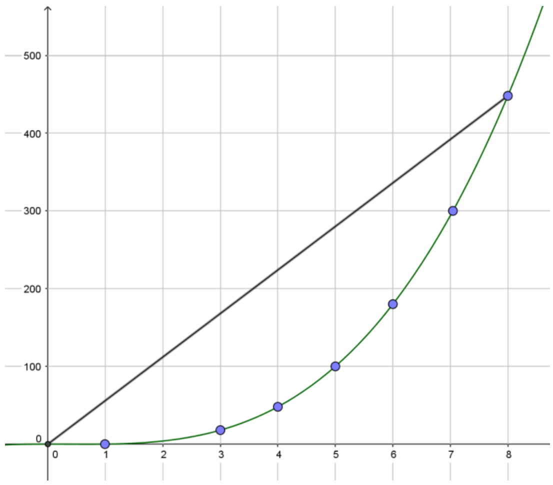
What are the implications for teaching in your field?
As a result of our meetings as a group and coming up with a meaningful and relevant problem, we all feel we have benefitted hugely from this lesson and its preparation. The thinking and discussions we had about this was invaluable as we all learned new teaching methods we can bring back to our own classrooms. We put a lot of work into planning our lesson meticulously and we feel that having this done well, keeps us alert and better equipped to spot students struggling with problems or students with good new ideas. The use of the in-class problem-solving time is perfect to facilitate this.

As a result of this lesson, students were much more prepared and equipped in order to understand differentiation in the following lessons. They also were comfortable with this entire topic as a result of introducing differential calculus in this manner.
The class teacher reported a noticeably greater understanding of calculus when exam style "Contexts and Applications" questions were encountered by this group of students.

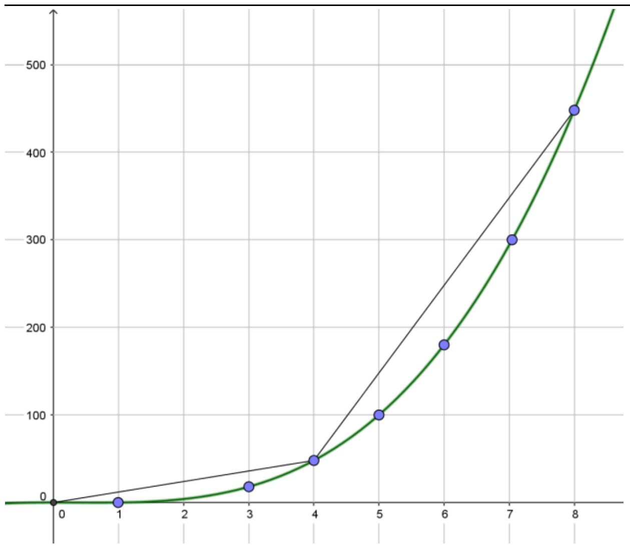
This problem-solving method was ideal for the teacher of the class, as it highlighted immediately mixed abilities and different problem areas for individual students. Because of these reasons, we feel this method of problem solving should be used in all of our classes more frequently where possible and that homework in the style of 'what did you learn' is invaluable as it highlights where the next class may begin.

Appendix A: 2 page worksheet given to students

Time (secs)	0	1	2	3	4	5	6	7	8
Distance(metres)	0	0	4	18	48	100	180	294	448

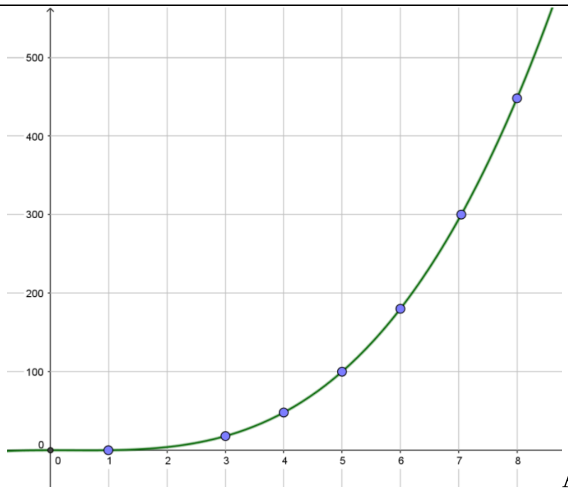


Avg. speed from 0 – 8s =



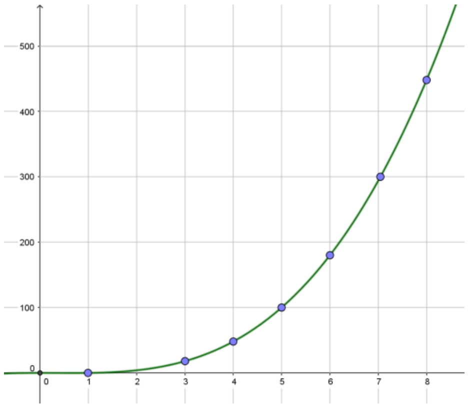
Avg. speed from 0 – 4s =

Avg. speed from 4 – 8s =

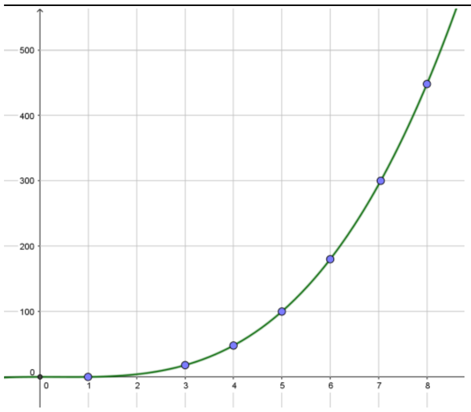


Avg. speed from 1 – 7s =

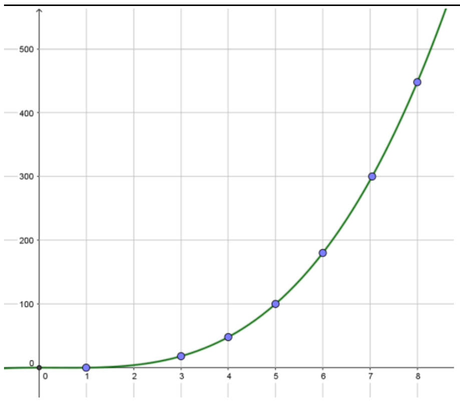
Time (secs)	0	1	2	3	4	5	6	7	8
Distance(metres)	0	0	4	18	48	100	180	294	448



Avg. speed from 2 – 6s =



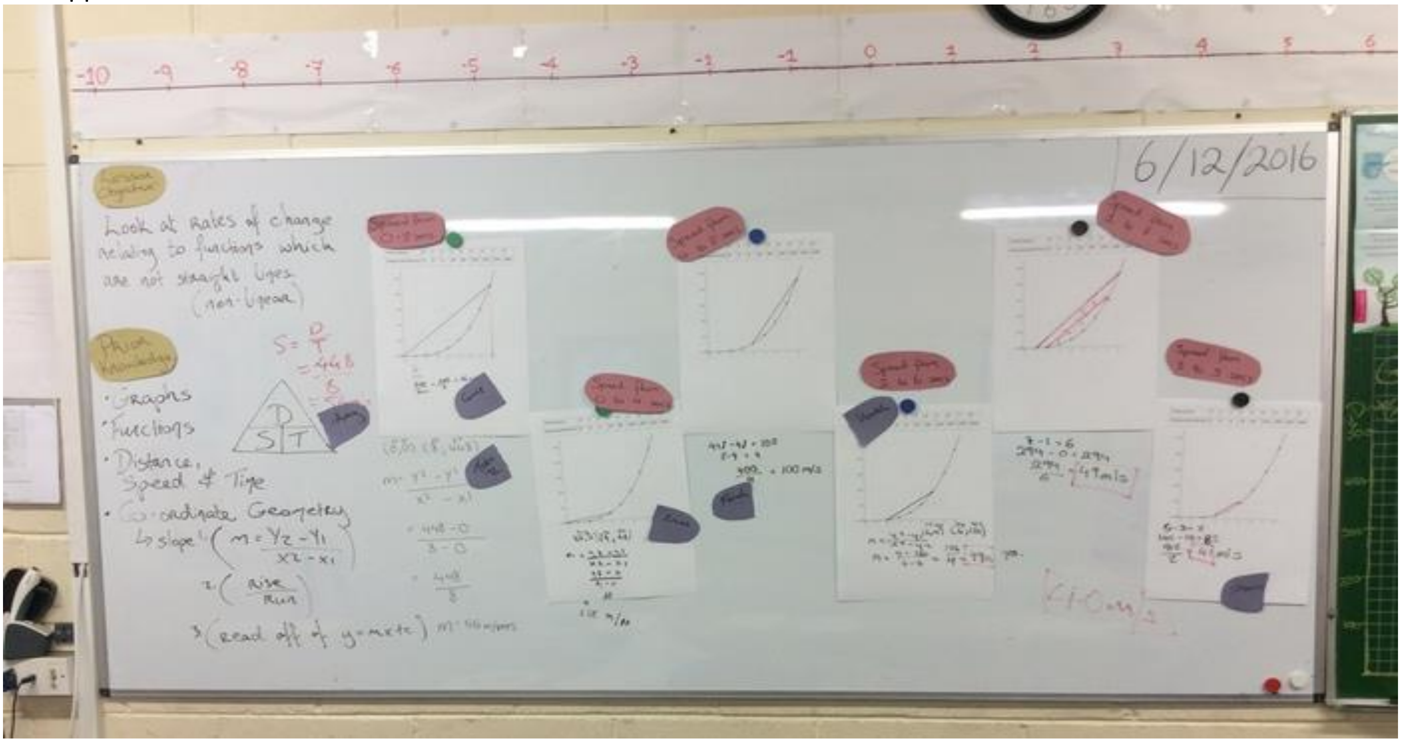
Avg. speed from 3 – 5s =



Our best estimate of the actual speed 4 seconds in is : ______________________________

This is based on : ____

Appendix B



Appendix C