# Lesson Research Proposal Introduction to Algebra

Common Level, First Years

For the lesson on 24 January 2018
At Clifden Community School, Ms Tierney's class
Teacher: Mary Tierney
Lesson plan developed by: Gráinne McGee, Mary Tierney, Roisin Leonard, James Levis

## 1. Title of the Lesson: *Box Bonanza*

## 2. Brief description of the lesson

During our lesson students will apply their learning from the previous problem solving lesson to find the solution to a problem they have not seen before. Initially students will be asked to find the number of matchsticks needed to make 5 adjacent squares. They will be encouraged to come up with their own mathematical expressions to represent their way of looking at the diagram and counting the sticks. They then will try to use their mathematical expressions to find out how many matchsticks they would need to make 20 squares. Students will be expected to figure out on their own that the numbers in their expressions can be generalised. Finally, students will be introduced to the concepts of a constant and variable. They will form an algebraic expression that arises from the pattern and will begin to see the use in replacing a variable with a letter.

## 3. Research Theme

In Clifden Community School we want our students to be[1]:

(a) Motivated to learn through having a clear sense of attainable and challenging learning outcomes, and,
(b) Able to take responsibility for their own learning, and use both the learning resources provided to them, and those that they resource themselves, to develop their skills and extend their knowledge.

For our students to be able to achieve these learner experiences, our teachers within our department must also strive individually. The focus for us, as teachers are to:

(a) Meaningfully differentiate content and activities in order to ensure that all students are challenged by the learning activities and experience success as learners
(b) Give students an opportunity to develop their own approaches to problem solving and hence take ownership of their own learning, and,
(c) Engage and involve students to be inclusive of all abilities and to get them all involved in the lesson.

## 4. Background & Rationale

Students struggle if they do not have any numerical values; they progress with difficulty through the study of relationships between quantities that are usually expressed using letters and symbols. For example: $2+5=7$ but $2+x = 2x$; expressions tend to be compressed incorrectly. Similarly, students who are able to calculate $(2+3)^2 = 25$ would often have a misconception that $(x+3)^2 = x^2+9$. Students have difficulties accepting that x can represent one value one day and a different value the next day.

[1] Looking at our School 2016 - A Quality Framework for Post-Primary Schools

Our current approach to introducing algebra is closely related to the content of the textbook that we use. We introduce letters to represent unknown values. In this project we hope to focus more on the meaning and purpose of variables, in particular, how variables can help us identify patterns and deal with algebraic expressions. We would like our students to understand that a variable can represent more than one value within the same context, and that the same letter can be used in different contexts.

After further discussion, it was decided among the group to take the entire section on 1st year algebra and design a unit beginning by adapting Takahashi's lessons[2] on linking numerical expressions to algebraic expressions and continuing up to and including the distributive laws for algebraic products of two binomials.

## 5. Relationship of the Unit to the Syllabus

| Related prior learning Outcomes | Learning outcomes for this unit | Related later learning outcomes |
|---|---|---|
| According to Primary School Mathematics Curriculum (pages 92-97), in the 6th class students:<br>• Know rules about brackets and priority of operations<br>• Identify square numbers and simple square roots<br>• Write whole numbers in exponential form<br>• Identify relationships and describe simple number patterns in words<br>• Explore the concept of a variable in the context of simple patterns, tables and simple formulae and substitute values for variables (*identify and discuss simple formulae from other strands e.g. d=2×r, a=l× w*).<br>• Translate word problems with a variable into number sentences | According to JC Mathematics Syllabus students should be able to:<br>• Devise strategies for computation that can be applied to any number. Students articulate the generalisation that underlies their strategy, firstly in common language and then in symbolic language (CIC, 3.1)<br>• Use tables and diagrams to represent a repeating pattern situation (CIC, 4.1)<br>• Generalise and explain relationships in words and numbers (CIC, 4.1)<br>• Write arithmetic expressions for particular terms in a sequence (CIC, 4.1)<br>• Examine relations derived from familiar or imaginary contexts or arrangements of tiles or blocks – and use tables, diagrams as a tool for analysing relations (CIC, 4.2)<br>• Develop and use their own mathematical strategies and ideas and consider those of others (CIC, 4.2)<br>• Present and interpret solutions, explaining and justifying methods | According to JC Mathematics Syllabus students should be able to:<br>• Solve linear equations in one variable<br>• Find the underlying formula algebraically for quadratic relations (HL)<br>• Show that relations have features (constant rate of change in linear and non-constant rate of change in quadratic relations) that can be represented in a variety of ways<br>• Recognise problems involving direct proportion<br>• Perform transformational activities: collecting like terms, simplifying expressions, substituting, expanding (4.6), working on expressions of the form:<br>$x^2 + bx + c$<br>$\frac{ax + by}{cx + dy}$<br>$(ax^2 + bx + c) \pm (dx^2 + ex + f)$<br>$(x \pm y)(x \pm y)$<br>$(ax + b)(cx^2 + dx + e)$<br>$(ax^2 + bx + c) \div (dx + e)$ |

[2] Akihiko Takahashi *Ways of counting and mathematical expressions*, Maths Counts 2016 http://www.projectmaths.ie/for-teachers/conferences/maths-counts-2016/

| | | |
|---|---|---|
| | (CIC, 4.2)<br><br>• Devise and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions (CIC, All Strands)<br><br>• Find the underlying formula for linear relations (4.3)<br><br>• Use letters to represent quantities that are variable (4.6)<br><br>• Perform transformational activities: collecting like terms, simplifying expressions, substituting, expanding (4.6) on such expressions as:<br><br>$ax + by$<br><br>$a(x + y)$<br><br>$axy$<br><br>$(ax + by + c) \pm (dx + ey + f)$<br><br>$a(bx + cy + d) + e(fx + gy + h)$<br><br>$(ax + b)(cx + d)$<br><br>• Connect graphical and symbolic representations of algebraic concepts<br><br>• Use real life problems as vehicles to motivate the use of algebra and algebraic thinking | Factorise expressions.<br><br>Solve 1st degree equations in two variables and quadratic equations. |

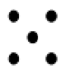
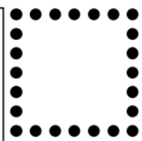
## 6. Goals of the Unit

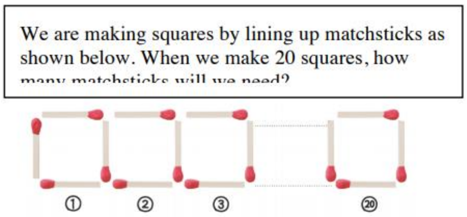
Our students will:

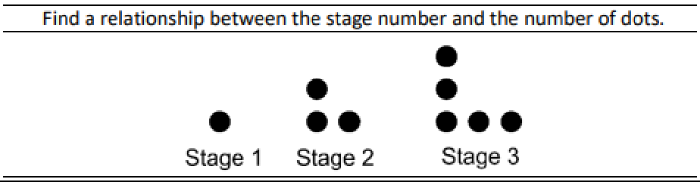
(a) Write simple mathematical expressions to express their ways of counting dots and infer other students' ways of thinking from their mathematical expressions.
(b) Understand that mathematical expressions for finding the number of dots in a diagram can be used to find the number of dots even without seeing the actual diagram (moving away from completing individual calculations towards developing algebraic expressions).
(c) Link numerical expressions to algebraic expressions (generalising) and understand the usefulness of algebraic expressions (efficient way of describing a relationship).
(d) Understand the meaning of a variable and constant.

(e) Identify algebraic expressions from worded problems using letters as variables.
(f) Write algebraic expressions omitting the multiplication sign (e.g. $2 \times a = 2a$, $1 \times a = 1a = a$, $(-1) \times a = -1a = -a$, $(-5) \times a = -5a$) and using exponents.
(g) Distinguish between $2a$ and $a^2$ (rectangular model can be used).
(h) See a purpose of substitution (problems in context can be used, e.g. an expression for the circumference/area of a circle, an expression that is used by police to find the minimum speed of a car by measuring the skid marks, an expression for converting Celsius to Fahrenheit, an expression for a speed of sound at a given temperature used to determine how far the lightning hit by measuring how long after a flash thunder was heard).
(i) Substitute different numerical values of the variable in an expression linking their knowledge of the order of operations in numerical sums to those in algebraic expressions (e.g. students should be able to find the values of $-a, a^2, -a^2, (-a)^2$ when $a$ is negative).
(j) Use a rectangular model to illustrate addition of like terms – terms that have the same letters (e.g. $6x + 3x = 9x$).
(k) Identify both the distributive and the commutative properties within the algebraic expressions.
(l) Use the array method to multiply monomial by binomial expressions and binomial by binomial expressions.

## 7. Unit Plan

<table>
<tr><th>Lesson</th><th>Learning goal(s) and tasks</th></tr>
<tr><td>1</td><td>Introduction to mathematical expressions<br>a) Students create numerical expressions to express their ways of counting, and<br>b) infer other students' ways of thinking from their mathematical expressions.<br>c) Students use the mathematical expressions they developed while counting the dots when there are 7 dots on each side of the square to determine the total number of dots when the number of dots on each side is changed.
<table>
<tr><td>Warm up task<br>Using the two diagrams shown on the right to help students see how mathematical expressions can be used to show ways of counting the number of dots.<br></td><td>Main problem<br>Think about ways to count the number of dots in the picture shown on the right.<br>For each way of counting, write a mathematical expression that describes each of your methods of<br></td><td>Expanding learning<br>Using a mathematical expression developed beforehand find the total number of dots when there are 10 dots on each side.</td></tr>
</table></td></tr>
<tr><td>2<br>Research Lesson</td><td>Introduction to variables and algebraic expressions<br>a) Link numerical expressions to algebraic expressions.<br>b) Introduce a variable (the number of squares varies, total number of matchsticks varies). Students identify the variables & constants.<br>c) Students use mathematical expressions which are originally described in words (e.g. total number of sticks=1+3×(number of squares)) and work up to bringing these to a general formula using letters.</td></tr>
</table>

<table>
<tr><td></td><td>


Note: Keeping the number of squares small initially, may help with this task.
<br>
Tabulate the results
<table>
<tr><th>No. of squares</th><th>Expression giving the total no. of matchsticks</th></tr>
<tr><td>1</td><td></td></tr>
<tr><td>2</td><td></td></tr>
<tr><td>3</td><td></td></tr>
<tr><td>□</td><td></td></tr>
<tr><td>20</td><td></td></tr>
<tr><td>□</td><td></td></tr>
<tr><td>$x$</td><td></td></tr>
</table>
</td></tr>
<tr><td>3</td><td>
Becoming fluent in mathematical expressions using letters to represent variables.


</td></tr>
<tr><td>4</td><td>
Follow up lesson.<br>
a) Students see algebraic expressions as an efficient way of describing a relationship.<br>
b) Students convert sentences into algebraic expressions using letters.<br>
c) Given a rectangle with side lengths $a$ cm and $b$ cm, students identify the quantities that can be represented by the algebraic expressions $a \times b$ and $2 \times a + 2 \times b$. Students find other quantities that can be represented by $c = a \times b$.
</td></tr>
<tr><td>5</td><td>
Multiplication<br>
a) Students write algebraic expressions omitting the multiplication symbol, e.g. $1 + 3 \times x = 1 + 3x, a \times 2 = 2a,\ \ 1 \times a = 1a = a, (-1) \times a = -1a = -a, (-5) \times a = -5a$<br>
b) Students identify the commutative properties in simple multiplication of algebraic expressions, e.g. $(5\mathrm{a})(4) = 5 \times \mathrm{a} \times 4 = 5 \times 4 \times \mathrm{a} = 20\mathrm{a}$. A misconception $3(7a) = 3(7) \times 3a$ is addressed.<br>
c) Use exponents (“How can we use algebraic expressions to represent an area of a square that has $a$ cm sides, or a volume of a cube that has $a$ cm long edges?”)<br>
d) Students distinguish between $2a$ and $a^2$.
</td></tr>
<tr><td>6</td><td>Addition of like terms</td></tr>
<tr><td>7</td><td>
Introduction to substitution<br>
Through solving problems in context, students see a purpose of substitution (see (h) in the Goals of the Unit).
</td></tr>
<tr><td>8</td><td>
Practicing substitution<br>
a) Students substitute numerical values for the variable in algebraic expressions, including finding the values of $-a, a^2, -a^2, (-a)^2$ when $a$ is negative.<br>
b) Students understand that the same rules apply as per “bimdas”, the order of operations.
</td></tr>
<tr><td>9</td><td>
Multiplying monomial by binomial expressions<br>
a) Students understand why $a(b + c)$ is equivalent to $ab + ac$.<br>
b) Apply the distributive law in simple expressions, e.g. $2(x + 3)$ or $5(2a - 5)$ or $3x(7 + 4x)$.<br>
c) Rectangular model is used. A common misconception that $2(x + 3) = 2x + 3$ is addressed.
</td></tr>
<tr><td>10</td><td>Practice multiplying a monomial by binomial.</td></tr>
<tr><td>11</td><td>Multiplying two binomial expressions</td></tr>
</table>

| | |
|---|---|
| | a) Students understand why $(a+b)(c+d)$ is equivalent to $ac+ad+bc+bd$.<br>b) Students use the distributive law to multiply two binomial expressions.<br>c) Rectangular model is used. |
| 12 | *Practice multiplying a binomial by binomial.*<br>Students combine their prior knowledge of exponents with new knowledge to expand $(x+3)^2$ .<br>A common misconception that $(x+3)^2 = x^2 + 9$ is addressed. |

## 8. Goals of the Research Lesson:

(a) Goals of the lesson

Students:

- Write mathematical expressions to express their ways of counting sticks
- Using the mathematical expressions that they developed when counting sticks for a smaller number of squares, students determine the number of sticks required for a larger number of squares.
- Recognise that there is a pattern in the given problem, and from that a formula can be generated.
- See the benefit of using formulas to calculate large quantities.
- Identify and differentiate between a constant and a variable.

(b) Key Skills & Statements Of Learning.

During the design of this lesson the Junior Cycle Key Skills and Statements of learning (24) were closely considered.

This lesson aims to include the following key skills:

- Managing information & thinking: Students are given time to work individually on the problem. They must interpret and analyse it by themselves and devise a variety of methods to solve the problem.
- Being creative: Students will be encouraged to present alternative solutions to the problem given.
- Communicating: Students will demonstrate their knowledge and explain the reasoning behind their solutions.
- Working with Others: Students will learn with and from each other.
- Being Literate: Students express their ideas to the class, explaining their thought process in a clear manner.
- Being Numerate: Students will represent words in a numerical fashion.

Statements of learning addressed: Student

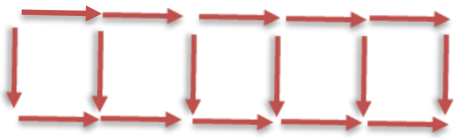
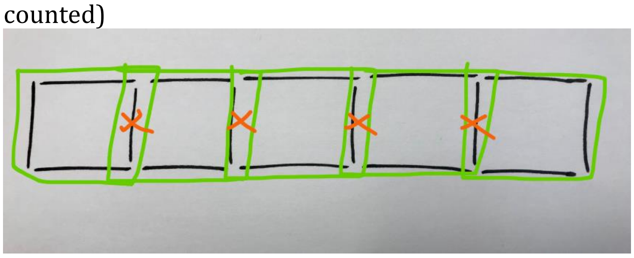
1. Communicates effectively using a variety of means in a range of contexts.

15. Recognises the potential uses of mathematical knowledge skills and understanding in all areas of learning.

16. Describes, illustrates, interprets, predicts and explains patterns and relationships.

17. Devises and evaluates strategies for investigating and solving problems using mathematical knowledge, reasoning and skills.

## 9. Flow of the Research Lesson:

| Steps, Learning Activities<br>Teacher's Questions and Expected Student Reactions | Teacher Support | Assessment |
|---|---|---|
| **Introduction**<br>The class recalls what they learned about mathematical expressions in the previous lesson. | Show selected pictures of previous lessons solutions. | Students show that they understand what is meant by writing a mathematical expression to express their way of counting. |
| **Posing the Task**<br><br>We are making squares by lining up matchsticks.<br><br>When we make 5 squares, how many matchsticks do we use? Write a mathematical expression to explain your method of counting." | The problem will be stated on the whiteboard and activity sheets will be handed out. | |
| **Student Individual Work**<br><br>***Anticipated responses***<br>Response 1: Student counts individual matchsticks 1+1+1+1…<br><br>Response 2: Possible misconception<br>5 squares each made out of 4 sticks: $5 \times 4 = 20$<br><br>Correct solution: $5 \times 4 - 4 = 16$ (5 squares and we see overlapping so we subtract the sticks that we double counted)<br><br><br>Response 3: 4+3+3+3+3=16 or $4 + 4 \times 3 = 16$ (1 square and all other squares have 3 sticks) | 10 minutes given for an individual work.<br><br>Teacher uses a seating chart to record each student's solutions and to prepare for the whole class discussion.<br><br>Students who finish early will be encouraged to come up with another expression.<br><br>If $5 \times 4 = 20$ is a commonly-held misconception it will be addressed at the start of the presentation of solutions. | |

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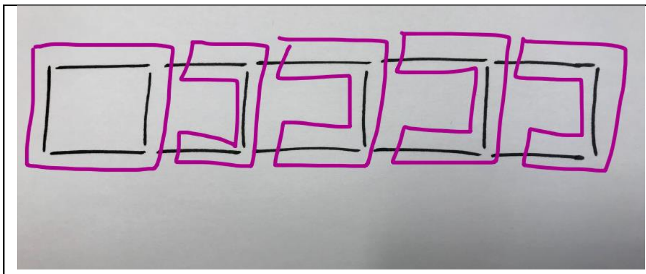
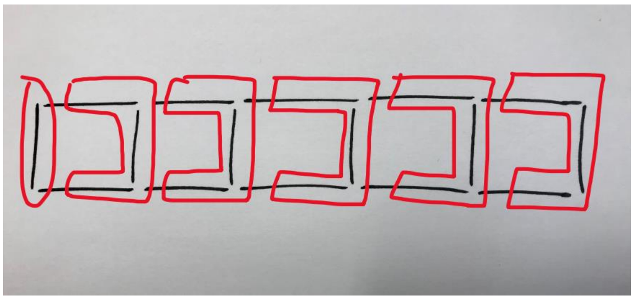
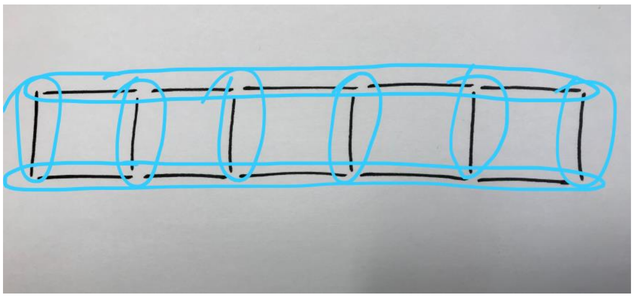
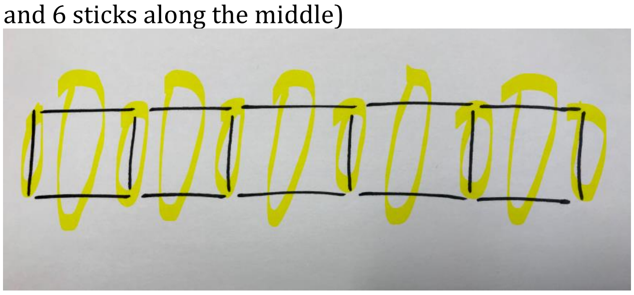
| <br>Response 4: 1+3+3+3+3+3=16 or $1 + 5 \times 3 = 16$ (1 stick and then all squares have 3 sticks)<br><br>Response5: $2 \times 5 + 6 = 16$ (set of 5 along the top, set of 5 along the bottom and 6 sticks along the middle)<br><br>Response 6: $5 \times 2 + 6 = 16$ (2 sticks – there are 5 sets of these<br>and 6 sticks along the middle)<br><br>Response 7. 3 x 4 + 2 x 3 | | |

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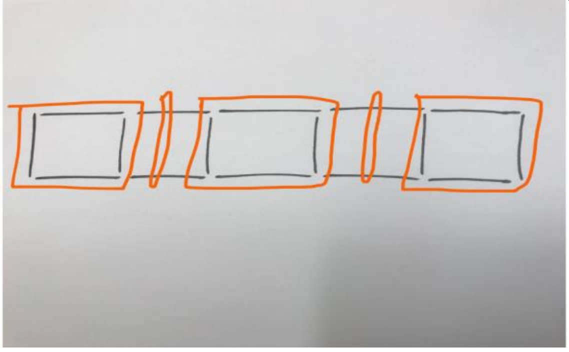
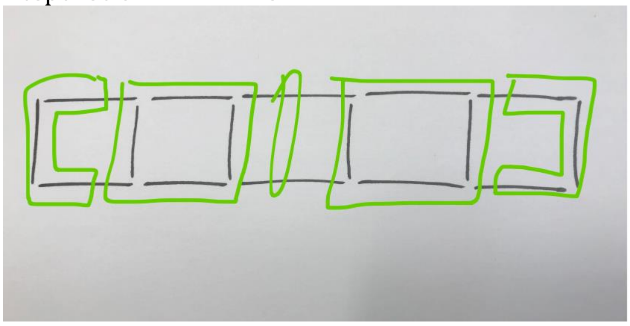
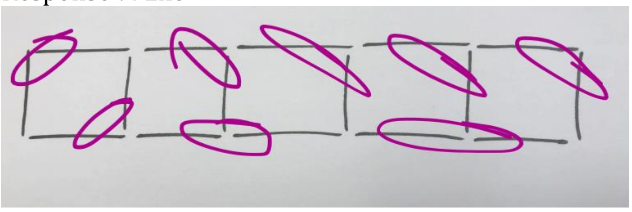
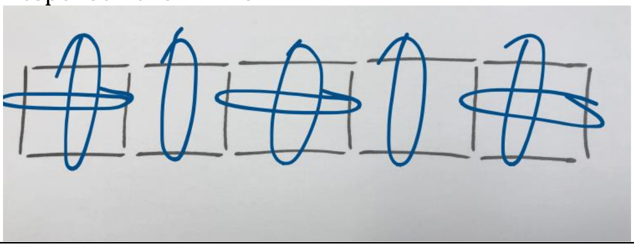
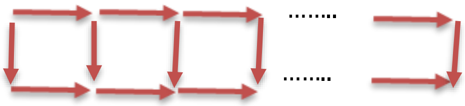
| <br>Response 8: 2 x 4 + 2 x 3 + 2<br><br>Response 9: 2x8<br><br>Response 10: 5 x 2 + 3 x 2<br> | | |
| **Ceardaíocht /Comparing and Discussing**<br><br>Invite a student who came up with an expression to come to the board and show his/her solution.<br><br>Let other students interpret that expression before the selected student explains it to the class.<br><br>Repeat these steps with every mathematical expression.<br><br>2) If $5 \times 4 = 20$ is a commonly-held misconception it will be addressed at the start of the presentation of solutions.<br><br>Did anyone else solve it the same way? Can you explain this method? Did we make a similar discovery yesterday with the dots? | If nobody came up with Solution 4, teacher proposes it as a solution coming from another class and asks students to explain how it can be visualised. It will be used later to introduce a variable $x$ as the number of squares that can vary. | Which expression relates to the number of boxes? |

<table>
<tr>
<td>………………</td>
<td>3) -4) Where is the 3? Show me.<br>5) -6) Is that 2 sets of 5 or 5 sets of 2?</td>
<td>Can you show me your expression on the diagram?</td>
</tr>
<tr>
<td>Posing extended task<br>“How many matchsticks would be needed to make 20 squares?<br>Use the expressions you have developed above to find your answer without drawing 20 boxes.”<br></td>
<td>Question posed on the whiteboard.<br><br>Students can write their answers on A4 sheets of paper or show me boards and hold them up to show to the teacher.</td>
<td></td>
</tr>
<tr>
<td>Anticipated responses<br><br>Response1. 4(19) + 3<br><br>Response 2) 20 × 4 – 19 or<br>4x20=80<br>80 -20 +1<br><br>Response 3) 4+19x3=61<br><br>Response 4) 1+20x3 (1 stick and 20 sets of 3)<br><br>Response 5) 20+20+21</td>
<td><br><br>So what does the 4 mean?<br><br>Do you see this?<br>20 = take one from each square<br>And add back the last 1<br><br>Why 20?<br><br>Which expression from earlier are you changing?</td>
<td></td>
</tr>
<tr>
<td>Discussing
<table>
<tr><th>No. of Squares</th><th>No. of Matchsticks</th></tr>
<tr><td>1 square<br>2 squares<br>3 squares<br>4 squares<br>5 squares<br><br>10 squares<br><br>20 squares</td><td>1+1(3)<br>1+2(3)<br>1+3(3)<br>1+4(3)<br>1+5(3)<br><br>1+10(3)<br><br>1+20(3) → stays same constant<br>↓ Stays same constant<br>↘ Changes variable</td></tr>
</table>
Do you notice any pattern in the numbers in these expressions?<br><br>Which number in the expression will be different if there are 50 squares? Which numbers are staying the same? (constant)<br><br>How would you read these expressions to someone who cannot see the board? Think of a single sentence that would deal with all the cases in the table.<br><br>“No. of matchsticks equals 1 plus the no. of squares</td>
<td>After the presentation of solutions to the extended task, Approach 4 will be selected and results tabulated starting with 5 and 20 squares and then expanding the table above and below.<br><br>Students show that they can use the expression efficiently.</td>
<td>Do you notice anything about the expressions?<br><br>Is there a pattern occurring here?<br><br>What do you call a number that stays the same/ doesn’t change? (constant)<br><br>Do you know a term for the number that keeps changing? (variable)<br><br>What can we use to represent the number that keeps changing? (x)</td>
</tr>
</table>

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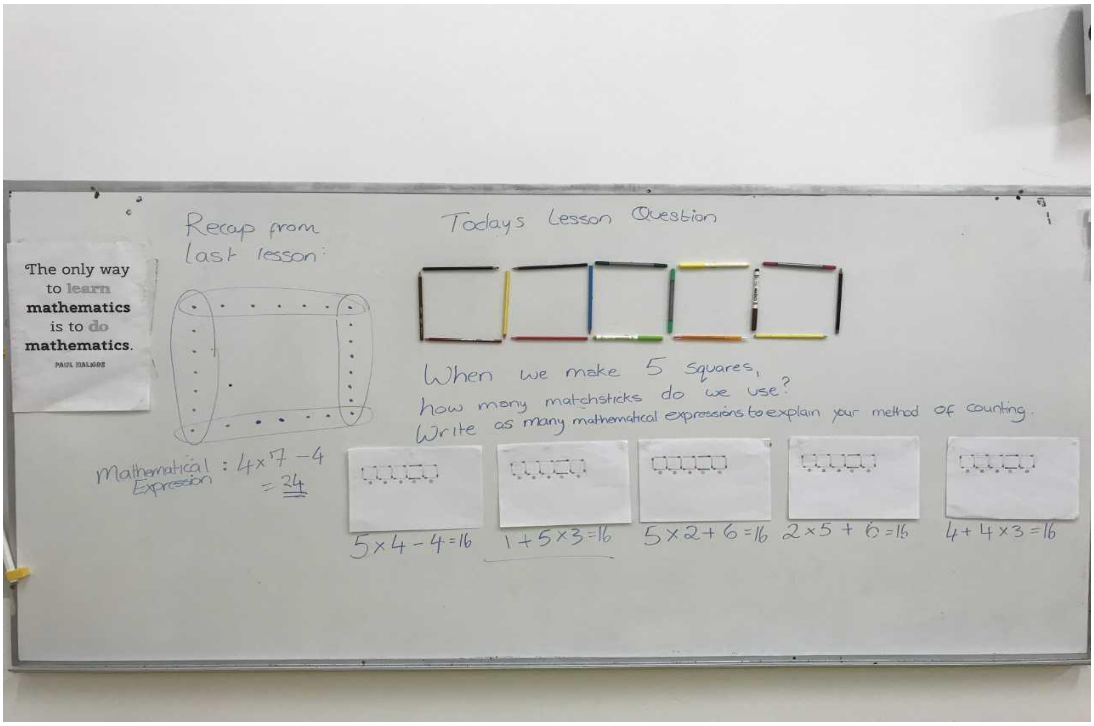
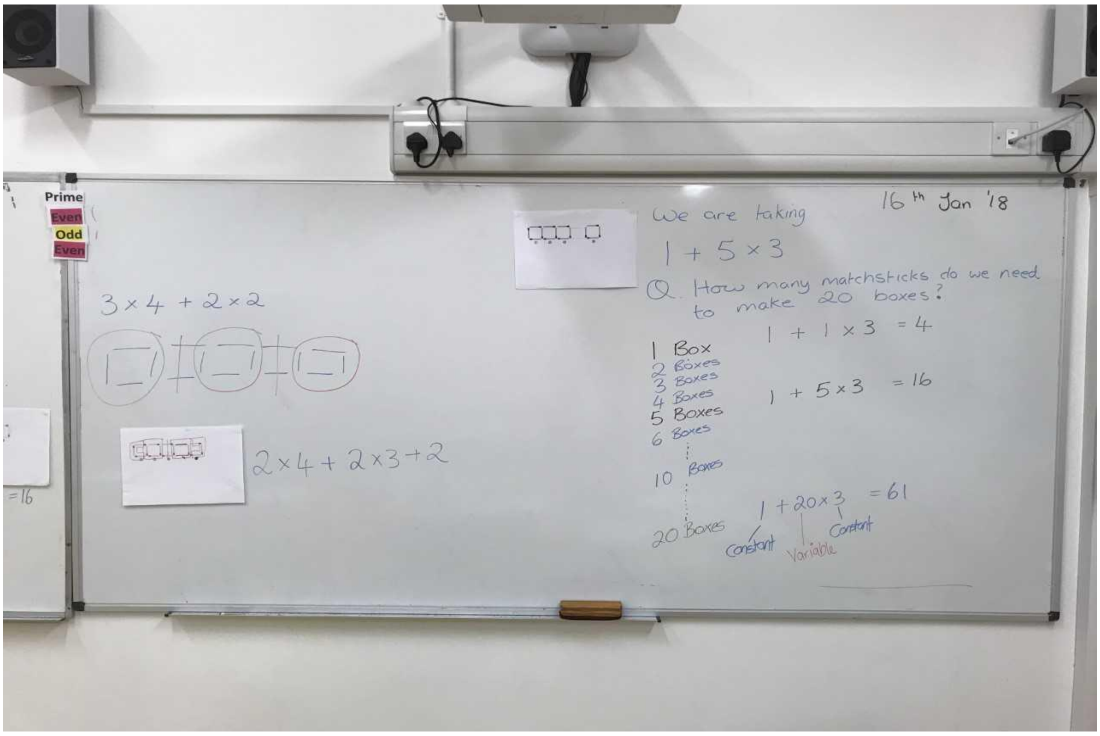
| times 3.” | | |
| **Summing up & Reflection**<br><br>We learned:<br><br>• How to formulate a mathematical expression by grouping the matchsticks. This allowed us to solve this problem ie. How many matchsticks are needed to make 5 boxes… Extension 20 boxes<br><br>• Mathematical expressions can be used to count large numbers more efficiently<br><br>• Meaning of the terms ‘Constant’ and ‘Variable’<br><br>Ask students to write a reflection. | | |

## 10. Board Plan






## 11. Evaluation

The general consensus was that the lesson was very successful, with the goals of the lesson achieved. The students were engaged in the task from the beginning, which was evident by their numerous attempts on the worksheet and tried to come up with a couple of different. The worksheet led to more than one solution being found and motivated the students to try attaining more solutions. The majority of students formed groups and wrote mathematical expressions getting the correct answer of 16 matchsticks.

During ceardaíocht, it was evident that students were able to write mathematical expressions to express their ways of counting sticks from the many responses that were obtained. Using the mathematical expressions that they developed when counting sticks for a smaller number of squares, students determined the number of sticks required for a larger number of squares. The students recognized that there is a pattern in the given problem, and from that a formula can be generated. From their work and reflection students were able to see the benefit of using formulas to calculate large quantities. The extension activity allowed students to use what they had already derived to solve a new problem and allowed the students to identify and differentiate between a constant and a variable.

### 12. Reflection

It was agreed by all that the task posed was straightforward and engaging for all students. The students used many different methods to solve the problem including a visual method; drawing the different ways they could see the 16 matchsticks. They drew circles to group the matchsticks which in turn helped them to write a mathematical expression. They used the idea of overlapping circles to help form similar mathematical expressions to the ones used in previous lessons. Some students went directly into writing the mathematical expressions before then trying to represent them visually on a diagram. Some students physically counted the number of matchsticks.

The students understood better initially when the previous lesson was revisited. Ms Tierney spoke about Monday's lesson where they had the problem with dots arranged in a square. The students remembered immediately. There was a change in understanding when the first answer was given, discussed and presented on the board, the class then understood better what exactly their answers should look like. A big change occurred when Ms Tierney switched the attention away from the confusion of the task sheet (the use of brackets instead of the multiplication symbol) to the board the class understood much better the table, mathematical expressions and how they could reach their answer.

There were some common misconceptions and misunderstandings, and these included; *brackets represent multiplication* - some students were confused with the final task sheet as there were brackets used instead of the multiplication symbol. *5x4=20* some students thought the answer was 20 straight away as they failed to see the overlap of matchsticks. *Mathematical expressions can only have 1 variable.* This was a misconception that was completely fair as they have not moved on to expressions with more than 1 variable.

Some comments or questions that students had were linked to their understanding to previous lessons; *"Can we use the same mathematical expressions we used for the dots during Monday's class"*. Students commented on what they saw using words and numbers. Some students wondered why they needed to learn about these mathematical expressions and where they could use them. Some students asked *if there was only one variable in each mathematical expression* and although this is not always the case, we felt that for their level of understanding that this would be addressed later. Students asked if the constant can ever change.

All students displayed a positive disposition, and were really engaged in the activity, and tried to come up with a couple of different solutions. They had to really think hard to come up with new expressions and they also had to focus when their classmates presented a solution to see if they could understand somebody else's work. One student wasn't sure about his own solutions, but his classmate helped him see his own answer was correct. All students made an effort to complete both tasks and were respectful and listened when their classmates were presenting.

**Recommendations:**
Put a mark beside the students work indicating that is the work to be shown when they are asked to come to the board. This will avoid any confusion.

BIMDAS –the importance of order of operations within the Mathematical Expression
eg. 1 + 5 X 3 = 16 ensuring you get the correct answer.