Lesson Research Proposal for 5th year Higher Level on Exponential Functions

For the lesson on date: 13/12/17
At Maynooth Education Campus
Instructor: Michelle Kelly
Lesson plan developed by: Michelle Kelly, Alice Mooney, Gráinne O' Rourke, Peter Lawlor

1. Title of the Lesson: It's Moore of a Process

2. Brief description of the lesson

Students investigate Moore's Law in order to apply their understanding of Logarithms and Exponential Functions and their relationship to their inverse.

3. Research Theme

At our school,

- We want our students to report on, present and explain the process and outcome of learning activities to a highly competent level.

- We want teachers to engage in planning for assessing all relevant aspects of students' learning using both assessment of learning and assessment for learning.

As a Mathematics department, we will actively support the achievement of these goals by endeavouring to do the following:

(a) Using Blooms Taxonomy and active questioning techniques, teachers will probe students to explain and provide rationale for their choice of methodologies and solutions to questions in class. Key Mathematical terminology will be highlighted throughout the lessons.

(b) Teachers will employ a number of Assessment for Learning techniques. For example, involving students in their own learning by assessing their own work and reflecting on their learning at the end of key lessons.

4. Background & Rationale

a) The relationship between functions and inverse functions is potentially difficult for students to grasp. Identifying the interchangeability from exponential form to logarithmic form can prove difficult and demanding for students in a 5th year higher level class when solving word problems or modelling.

b) From the chief examiners report it was highlighted that while problem-solving in unfamiliar contexts is an important skill, it cannot be achieved unless students are competent in the basics of the syllabus. It was found that at higher level candidates struggled in particular with a problem-solving question. While the question was a challenging one, there were a number of different possible approaches to solving the problem. However, candidates showed little initiative in coming up with a solution.

5. Relationship of the Unit to the Syllabus

Related prior learning Outcomes	Learning outcomes for this unit	Related later learning outcomes
From JC: Inverse operations Exponential functions	• Graph functions of the form	**Differentiation** • Differentiate the following functions

From LC: Exploring the inverses of linear and quadratic functions using algebra and graphing.	- ab^x where a ∈ N, b, x ∈ R - ab^x where a, b ∈ R - logarithmic - exponential • Interpret equations of the form $f(x) = g(x)$ as a comparison of the above function • Use graphical methods to find approximate solutions to - $f(x) = 0$ - $f(x) = k$ $f(x) = g(x)$ where $f(x)$ and $g(x)$ are of the above form, or where graphs of $f(x)$ and $g(x)$ are provided. • Communicate mathematics verbally and in written form • Apply their knowledge and skills to solve problems in familiar and unfamiliar contexts • Analyse information presented verbally and translate it into mathematical form • Devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.	- exponential - inverse functions - logarithms • Find the derivatives of sums, differences, products, quotients and compositions of functions of the above form • Apply the differentiation of above functions to solve problems **Financial Maths** • Solve problems involving finite geometric series including applications such as financial applications, e.g. deriving the formula for a mortgage repayment • Use present value when solving problems involving loan repayments and investments

6. Goals of the Unit

a) Students will revise the rules of logarithms and exponentials.

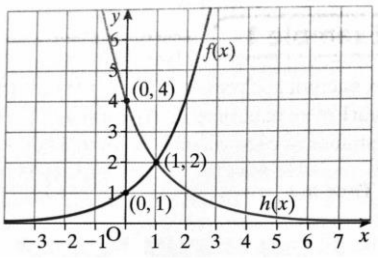
b) Students will be capable of graphing logarithmic and exponential functions.

c) Students will understand the inverse relationship between logarithmic and exponential functions.

d) Ability to convert a word problem to an equivalent mathematical equation.

e) Students to be willing to approach with confidence problems involving exponential and logarithmic functions.

7. Unit Plan

Lesson	Learning goal(s) and tasks
1	Students will revise indices rules by skills practice. • Solve problems using the rules for indices (where a, b ∈ R; p, q ∈ Q; $a^p, a^q \in$ Q; a, b ≠ 0): - $a^p a^q = a^{p+q}$ - $\frac{a^p}{a^q} = a^{p+q}$ - $a^0 = 1$ - $(a^p)^q = a^{pq}$ - $a^{\frac{1}{q}} = \sqrt[q]{a} \;\; q \in Z, q \neq 0, a > 0$ - $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p \;\; p, q \in Z, q \neq 0, a > 0$ - $a^{-p} = \frac{1}{a^p}$ - $(ab)^p = a^p b^p$ - $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$
2	Students will revise Logarithmic rules by skills practice. • Solve problems using the rules of logarithms - $\log_a(xy) = \log_a x + \log_a y$ - $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ - $\log_a x^q = q \log_a x$ - $\log_a a = 1 \;\; and \;\; \log_a 1 = 0$ - $\log_a x = \frac{\log_b x}{\log_b a}$
3	Students will graph exponential functions and solve simple mathematical problems relating to exponential graphs. For example: *The graphs of two exponential functions, y=Ab^x are given in this diagram find the value of A and b for each graph* 
4	Students will graph logarithmic functions and solve simple mathematical problems relating to logarithmic graphs. For example:

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Consider the function $y = \log_3 x$.
(i) Complete the following table.
<table>
<tr><td>x</td><td>$\frac{1}{9}$</td><td></td><td>1</td><td>3</td><td></td></tr>
<tr><td>$y = \log_3 x$</td><td></td><td>−1</td><td></td><td></td><td>2</td></tr>
</table>
(ii) Using the values in this table, sketch the graph of $y = \log_3 x$.
(iii) Estimate the value of $\log_3 2.5$ from your graph.
(iv) Using the change of base rule, $\log_3 x = \frac{\log_{10} x}{\log_{10} 3}$, find the value of $y = \log_3 2.5$.
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Students will understand the inverse relationship between logarithmic and exponential functions using algebraic and graphical methods.

In this lesson, we formalise this graphical observation with the idea of inverse functions. Students have not yet been exposed to the idea of an inverse function. In order to clarify the procedure for finding an inverse function, we start with algebraic functions before returning to logarithms and exponential functions. The concept will be introduced using questions in the style of:

A. Solve an equation of the form $(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

B. For exponential models, express as a logarithm the solution to $ab\ ct = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm.
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The Research Lesson</td>
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Link exponential and logarithm knowledge to solve word problems, and through this develop a deeper understanding of the relationship between functions and their inverse.
• Students introduced to a scenario involving generating a formula for Moore’s Law. Students are given the general form of the exponential function and given two examples of values and their respective times in order to complete the function.
• Students are encouraged to use a variety of methods to solve questions involving the use of the function.
• Students are asked to find the inverse of the exponential function and use it to find the relationship between both functions.
Students are further pushed to explore their understanding of the mathematics in the context of this problem through class discussion and questioning.
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<td>Use exponential and logarithm rules to solve word problem questions.</td>
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8. Goals of the Research Lesson:

a) Students will approach with confidence word problems involving exponential and logarithmic functions.
b) Ability to convert a word problem to an equivalent mathematical equation.
c) Ability to manipulate formula to isolate the given variable.
d) Students will apply the conversion formulae from exponential to logarithmic form and vice versa, as required in the question.

e) Students will develop a deeper understanding of the relationship between functions and their inverse.

9. Flow of the Research Lesson:

Steps, Learning Activities Teacher's Questions and Expected Student Reactions	Teacher Support	Assessment
Introduction https://www.youtube.com/watch?v=To4qrCuwNDU	Putting Moore's law into context with a short video.	
Posing the Task **Question** Moore's Law states that processor power (P) for computers grows exponentially according to the model $P = a2^{bt}$ where a and b are constants and time t is measured in years. In 1965, the processing power of a computer was 2MHz. In 1967 the processing power increased to 4 MHz. Using as many methods as possible: a) Find the values of the constants a and b. b) What was the processing power in 1975? c) Find in what year the processing power increased to 524,288MHz. d) Find the inverse function for Moore's Law and explain the relationship between the function and its inverse.	Problem will be handed out on a worksheet to each student	Check that each student understands the key terms in the question.
Student Working on the problem **Solution** a) Find the values of the constants a and b. **Method 1: Algebraically** $P = a2^{bt}$ where a and b are constants and time t is measured in years. We know that in 1965 at $t = 0, P = 2$MHz $\Rightarrow P = a2^{bt}$ $\Rightarrow 2 = a2^{b(0)}$ $\Rightarrow 2 = a(1)$ $\Rightarrow 2 = a$ Hence, as $a = 2$, $P = 2(2^{bt})$ We also know that in 1967 at $t = 2, P = 4$MHz $\Rightarrow P = 2e^{bt}$ $\Rightarrow 4 = 2(2^{2b})$ $\Rightarrow 2 = 2^{2b}$ $\Rightarrow 2^1 = 2^{2b}$ $\Rightarrow 1 = 2b$ Equate powers $\Rightarrow b = \frac{1}{2}$ Hence, the model which illustrates Moore's Law is given by $P = 2\left(2^{\frac{1}{2}t}\right)$ **Solution** a) Find the values of the constants a and b. **Method 2: Trial and Error** $P = a2^{bt}$ where a and b are constants and time t is measured in years. We know that in 1965 at $t = 0, P = 2$MHz $\Rightarrow P = a2^{bt}$ $\Rightarrow 2 = a^{b(0)}$ $\Rightarrow 2 = a^1$ $\Rightarrow 2 = a$ Hence, as $a = 2$, $P = 2(2^{bt})$	Students will be asked to solve the problems in as many ways as possible. Think, pair, share will be employed. **Misconceptions:** *(a) Method 1* • *Inserting the incorrect values for the time. For example in the first instant stating t=1 and for the second value taking t=1 or 2 depending on the previous value for time.* • *Finding the incorrect value for $a^{b(0)}$, i.e. 0.* • *Calculating the incorrect value for*	Teachers circulate the room assessing students' work to plan how to orchestrate the presentation of students' work on the board and class discussion.

Solution

a) Find the values of the constants a and b.

Method 2: Trial and Error

We also know that in 1967 at $t = 2, P = 4$MHz

Substitute in $b = 1$

$\Rightarrow P = 2(2^{bt})$ $\Rightarrow 4 = 2(2^{1(2)})$ $\Rightarrow 4 \neq 2(4)$ $\Rightarrow 4 \neq 8$ ✗

b must be between 0 and 1, so try $b = \frac{1}{2}$

$\Rightarrow P = 2(2^{bt}) \Rightarrow 4 = 2(2^{\left(\frac{1}{2}\right)(2)}) \Rightarrow 4 = 2(2^1) \Rightarrow 4 = 4$ ✓

Hence, the model which illustrates Moore's Law is given by $P = 2(2^{\frac{1}{2}t})$.

Solution

b) What was the processing power in 1975?

Method 1: Sequencing method

Double the processing power every 2 years.

Year	Processing Power
1965	2
1967	4
1969	8
1971	16
1973	32
1975	64

Solution

b) What was the processing power in 1975?

Method 2(a): Table method/ substitution

Create a table for the function in the domain $0 < t \leq 10$, in 2 year steps.

Year	t	$P= 2(2^{\frac{1}{2}t})$	$P = 2(2^{\frac{1}{2}t})$
1965	0	$P= 2(2^{\frac{1}{2}(0)})$	2
1967	2	$P= 2(2^{\frac{1}{2}(2)})$	4
1969	4	$P= 2(2^{\frac{1}{2}(4)})$	8
1971	6	$P = 2(2^{\frac{1}{2}(6)})$	16
1973	8	$P = 2(2^{\frac{1}{2}(8)})$	32
1975	10	$P = 2(2^{\frac{1}{2}(10)})$	64

Solution

b) What was the processing power in 1975?

Method 2(b): Using the Table function on the calculator:

Create a table for the function in the domain $0 < t \leq 10$, in 2 year steps.

Year	t	$P = 2(2^{\frac{1}{2}t})$
1965	0	2
1967	2	4
1969	4	8
1971	6	16
1973	8	32
1975	10	64

b. Possibly stating that b=2, as students have not come across values of b which are fractions before.

(a) Method 2

- *Inserting the incorrect values for the time. For example in the first instant stating t=1 and for the second time stating that t=2.*
- *As a result finding 2 simultaneous equations which they find difficult to solve.*

(b) Method 1

- *Doubling the processing power for each year.*
- *Hence, getting 1965 = 2MHz, 1966 = 4MHz, 1967 = 8MHz and so on!*

(b) Method 2,(a), (b)

- *Inserting the incorrect values for t.*
- *Reading the incorrect value of the graph.*

(b) Method 3

- *Inserting the incorrect values for t.*

(c) Method 1

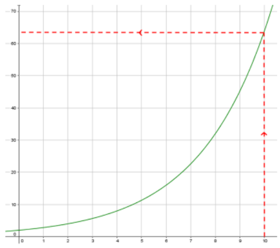
- *Calculation errors – simple manipulation errors*
- *Incorrect substitution into the exponential to logarithm conversion formula – Stop here, as they don't remember to*

Solution

b) What was the processing power in 1975?

Method 2: Draw the function in the domain $0 < t \leq 10$.

From the graph or the tables, we can see that: $t = 10$ years from the original time
$P = 2^{10} \cong 64$ MHz
Hence, the processing power in 1975 is 64MHz.



Solution

b) What was the processing power in 1975?

__Method 3:__

Using the formula: $t = 10$ years from the original time

$P = 2(2^{\frac{1}{2}t})$

$P = 2(2^{\frac{1}{2}(10)})$

$P = 2(2^{(5)})$

$P = 64$ MHz

Solution

c) Find in what year the processing power increased to 524,288MHz?

$P = 2(2^{\frac{1}{2}t})$ ...substitute 524288 in for P

$\Rightarrow 524{,}288 = 2(2^{\frac{1}{2}t})$ ...divide across by 2

$\Rightarrow \frac{524288}{2} = (2^{\frac{1}{2}t})$

$\Rightarrow 262144 = (2^{\frac{1}{2}t})$ ...change from exponential form into logarithm form using the conversion formula $a^x = y \Leftrightarrow \log_a y = x$

$\Rightarrow \log_2 262144 = \frac{1}{2}t$

$\Rightarrow 18 = \frac{1}{2}t$ ...multiply across by 2

$\Rightarrow 36 = t$

Hence, the processing power increased to 524,288MHz in 2001 (1965+36 years)

Solution

d) Find the inverse function for Moore's Law and explain the relationship between the function and its inverse.

Method 1: Algebraically

$P = 2(2^{\frac{1}{2}t})$ ...swap the variables P and t

$\Rightarrow t = 2(2^{\frac{1}{2}P})$ ...rearrange the formula to find P in terms of t

$\Rightarrow \frac{t}{2} = (2^{\frac{1}{2}P})$ ...change from exponential form into logarithm form using the conversion formula: $a^x = y \Leftrightarrow \log_a y = x$

$\Rightarrow \log_2\left(\frac{t}{2}\right) = \frac{1}{2}P$ ...multiply across by 2

$\Rightarrow 2\log_2\left(\frac{t}{2}\right) = P$

$\Rightarrow 2\log_2\left(\frac{t}{2}\right) = P^{-1}(t)$ is the inverse function

Solution

d) Find the inverse function for Moore's Law and explain the relationship between the function and its inverse.

Method 2: Swap the x and y coordinates of the couples.

Year	t	$P^{-1}(t)$
1965	2	0
1967	4	2
1969	8	4
1971	16	6
1973	32	8
1975	64	10

use the conversion formula

- *Finding the incorrect year as a result of a simple addition error*

(d) Method 1

- *Calculation errors – simple manipulation errors*
- *Failure to swap variables at the end of the process*
- *Failure to substitute $P^{-1}(t)$ at the end of the process*

(d) Method 2

- *Simple swapping errors*

(d) Method 3

- *Error in drawing the line y=x and hence the corresponding inverse function being inaccurate*

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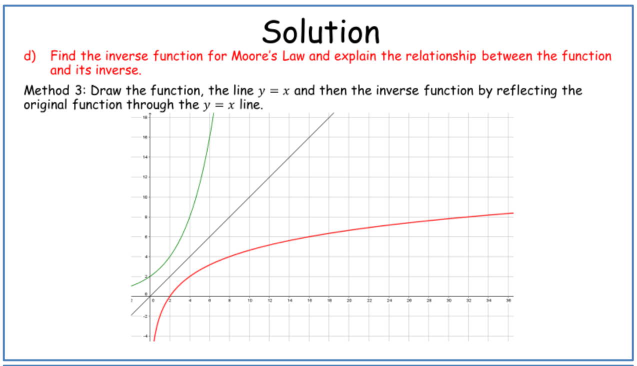

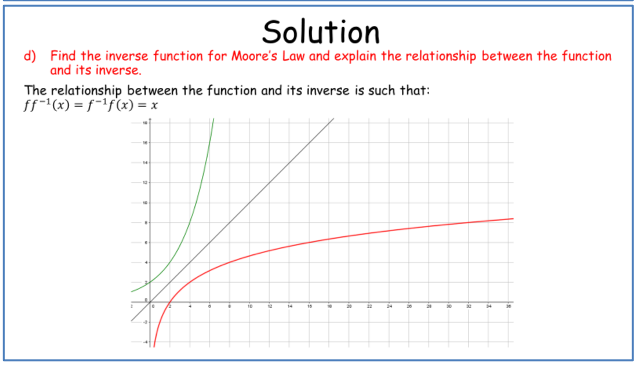

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Ceardaíocht /Comparing and Discussing
Focusing on section (d) of the question, the teacher probes and develops their ideas around the relationship between the function and its inverse. Also questioning why, the inverse of the function is useful and its real world application.

The teacher will ask the students to now look at second question

Question

Scientists have determined that a particular virus grows exponentially according to the model $\boldsymbol{V = ab^t}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants and time $\boldsymbol{t}$ is measured in weeks. Initially the virus has affected 32 people. In one week, the number of people infected rises to 48 people.
a) Find the values of the constants a and b.
b) Calculate the number of people infected by this virus within 7 weeks.
c) After how many weeks would the number of people infected by the virus increase to 1052.
d) To negate the growth of the virus, scientists develop a vaccine. What model would describe the path best suited for the vaccine?
<u>Use as many methods as possible!</u>
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Effective questions to include:

“What do you think”? (ask another student(s) other than the presenter)

“Why is that”? (Looking for evidence)

“Can you explain, in the current context, why our function did not equal zero at any time?”

“Did anyone else solve it the same way? Can you explain this method?”
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Teacher is looking for students to develop a stronger understand of the relationship between the function and its inverse by seeing if

The students drawing links between the two questions.

Supporting their answers with mathematical reasoning.

Engaging in the discussions about the scenario.
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Summing up & Reflection

Students are asked to further explore the ideas of the lesson through the homework activity (2nd
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The teacher will use the layout of the board work to help provide students with
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Using post-it notes surveys students will write down one
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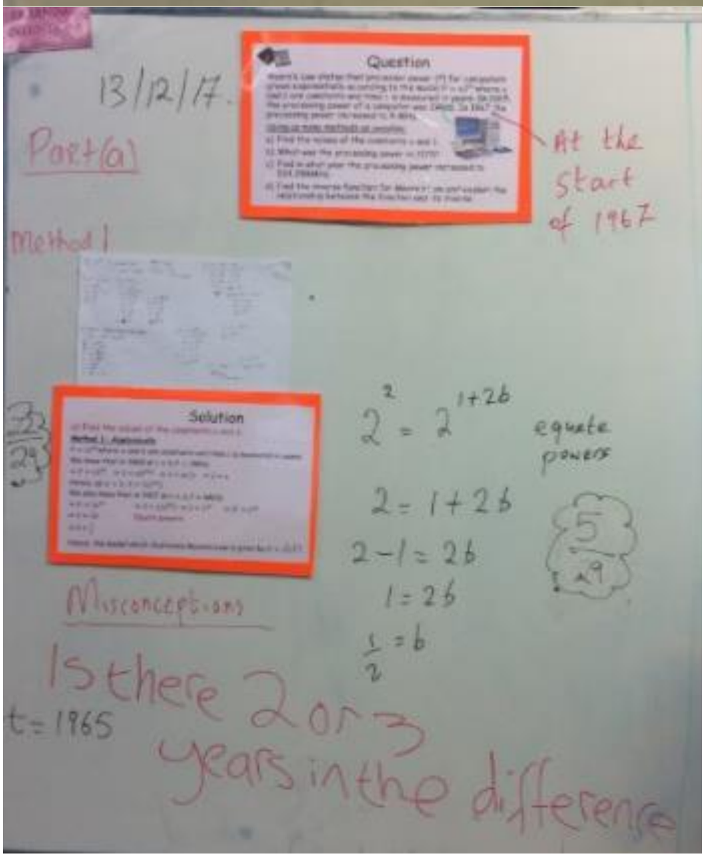
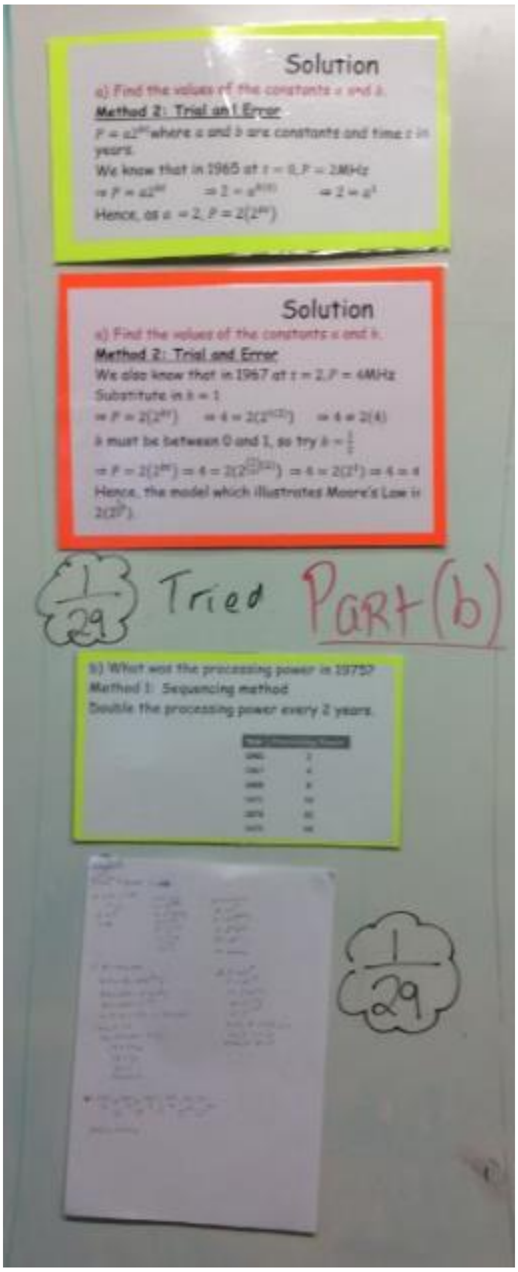
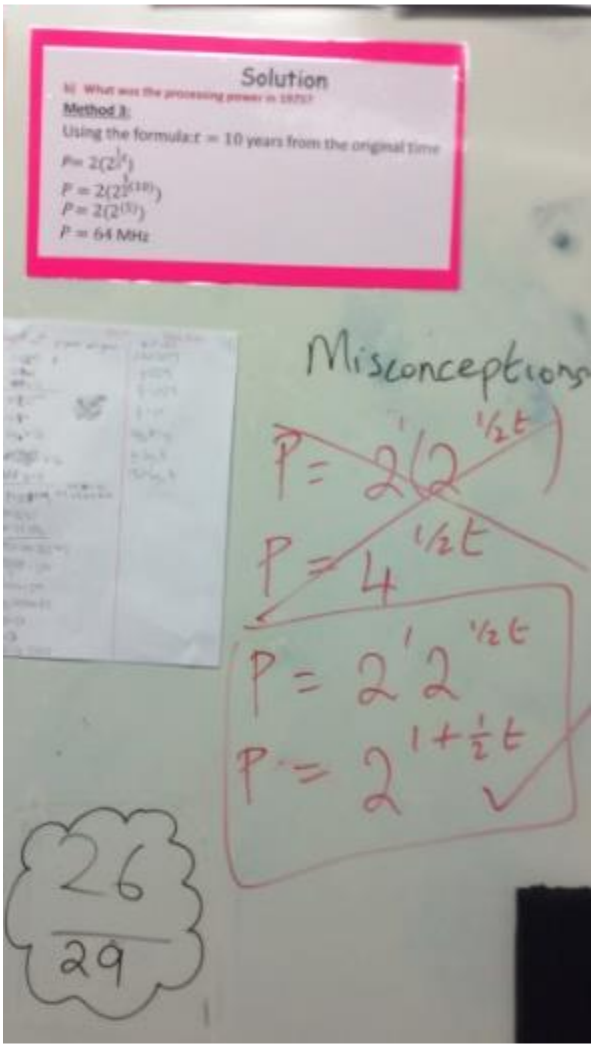
question) Students are asked to reflect on today's lesson.	a summary of the progression in their learning.	thing that they thought they did well today in class, one new thing they noticed, any questions they have still at the end of class.

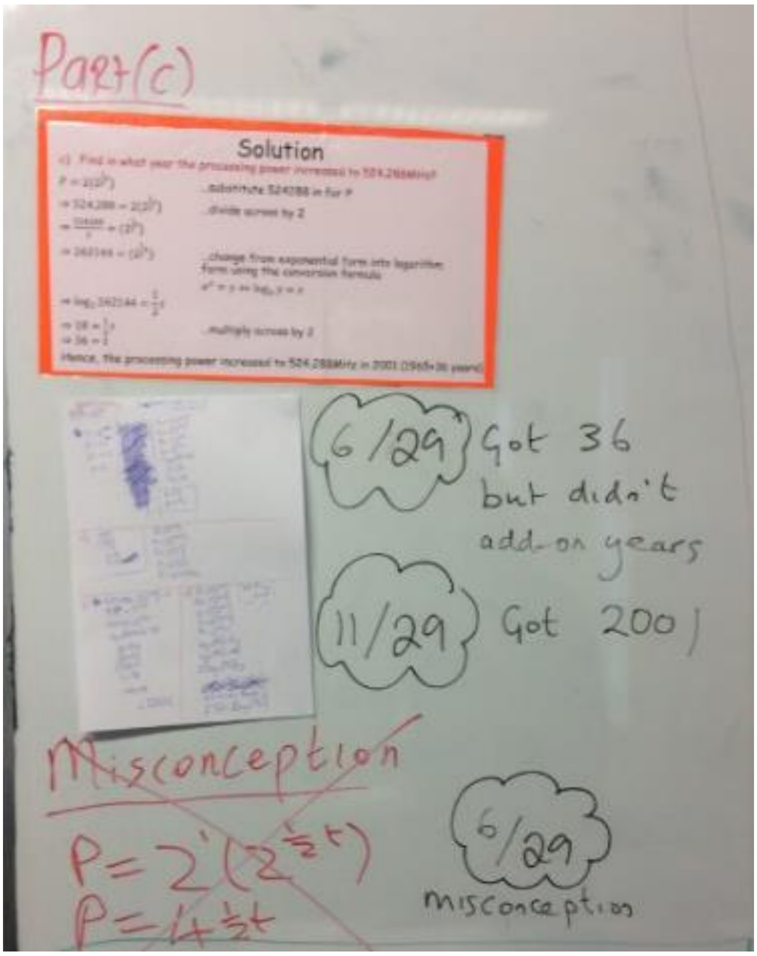
10. Board Plan



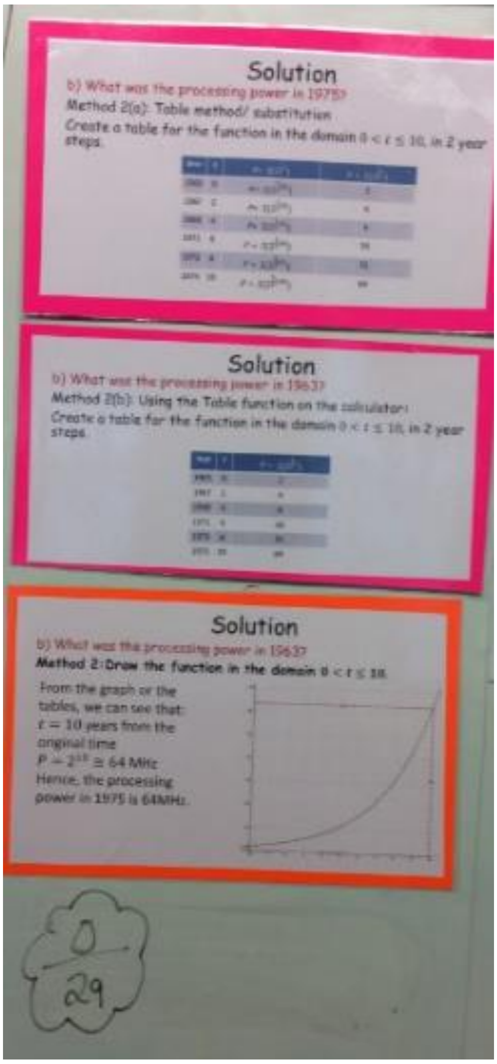




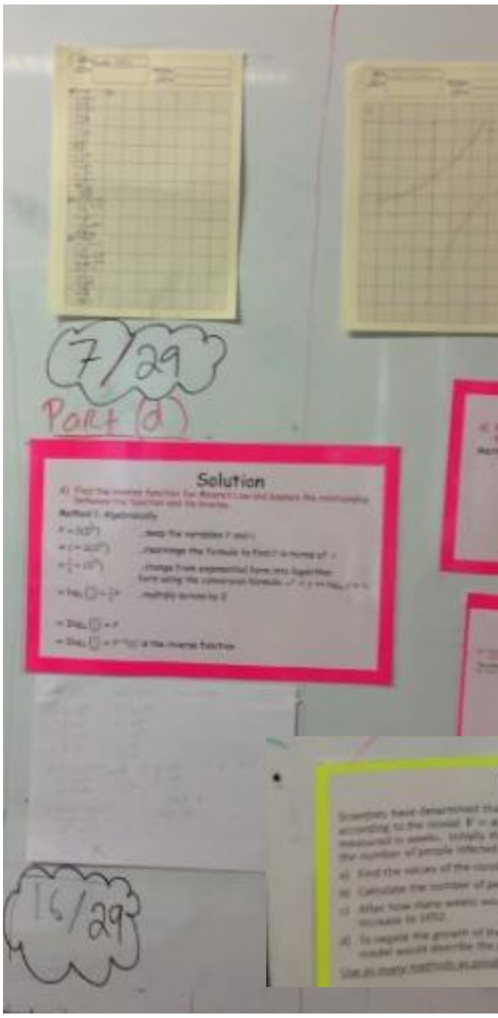
Solution
b) What was the processing power in 1975?
Method 3:
Using the formula: t = 10 years from the original time
P = 64 MHz
Misconceptions
26/29



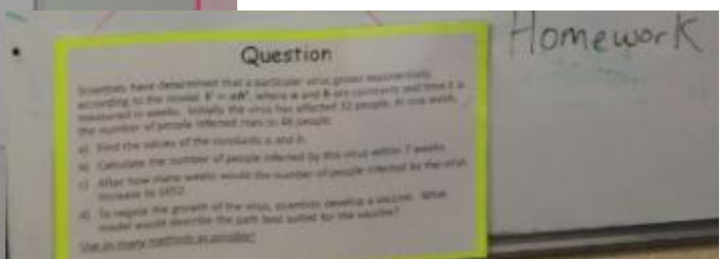
Part (c)
Solution
6/29
Got 36 but didn't add on years
11/29
Got 2001
Misconception
6/29
misconception



Solution
b) What was the processing power in 1975?
Method 2(a): Table method/ substitution
Solution
Method 2(b): Using the Table function on the calculator:
Solution
Method 2: Draw the function in the domain
From the graph or the tables, we can see that:
Hence, the processing power in 1975 is 64MHz.
0/29



7/29
Part (d)
Solution
16/29



Question
Homework

11. Evaluation

- Students reported on, presented and explained their solutions to the activity to a highly competent level.

- Approaches used by the students: Students used a number of approaches including algebraic methods, graphical methods and sequencing methods to solve the questions. These solutions were pre-empted by the team with the exception of two new methods used by students.

- Teachers involved engaged in assessment of students learning using both assessment of learning and assessment for learning.

- Teachers probed students to explain and provide rationale for their choice of methodologies and solutions to questions in class. Key Mathematical terminology was highlighted throughout the lesson.

- Teachers employed a number of Assessment for Learning techniques. For example, involving students in their own learning by assessing their own work and reflecting on their learning at the end of key lessons.

12. Reflection

The team hoped that all students would attempt all parts and try find a number of different methods per part and this was achieved by students

The introduction to the activity was well explained and the YouTube video used enhanced the real life application of exponentials.

Students engaged well with the activity overall and understood the goal of the lesson. Feedback from students included:

What did I do well?	**What did I learn?**
Learned from my mistakes	Learned to use different methods x 10
Parts (a),(b), (c) correct x 3	I learned that in part I my answer was wrong due to a misconception x 5
I used a method not thought of by the creators of the question x 3	I learned about the table method of doubling x 2
I did well in the first 2 parts of the question x 5	I used trial and error to find the answer because I didn't know what method to use
I managed my timing well x 2	I learned about Moore's Law x 3
I understood the question	I learned how Maths is used to improve technology
Algebraic methods x 3	I made mistakes that I won't make again
Used the Indices rules well	I found out a simpler way to find the inverse function by using a graph x 2
Group work x 2	Just swap the coordinates
I made mistakes but I managed to use my head to keep going	You put t = 0 at the start always

I got all parts correct	I didn’t expect to remember the method and I made mistakes with the Indices rules
Attempted all questions x 3	I go for the easiest method
I think outside the box	To read the question properly
I tried different methods	
I found the inverse of the function	

This feedback highlights the need for a problem solving class to be incorporated in schemes of work to encourage students to evaluate their own learning and to draw attention to any misconceptions students have.

Improvements:
Question:

- ‘In 1965’ and ‘In 1967’ should have been phrased as ‘At the beginning of 1965’ and ‘At the beginning of 1967’.
- The number of parts included in the question was too long for the time allocated.
- Perhaps could have limited the number of possible solutions per part (i.e. if the students do not come up with a solution then do not display this solution.)

Ceardaíocht question was well understood by students having completed the activity in class.

Benefits of participating in Lesson Study

One of the main benefits of lesson study is working with colleagues. In a busy school, time is always an issue. However, lesson study gave us the opportunity to use our CPD to have conversations about our students’ misconceptions in Maths and their problems with particular topics. Lesson study also provided us with the platform to work together to prepare a lesson designed specifically to challenge our students to work outside their comfort zone and apply their prior knowledge to solve a problem. Finally and perhaps most importantly, our students benefited. Being able to understand and solve the question boosted students confidence and morale.