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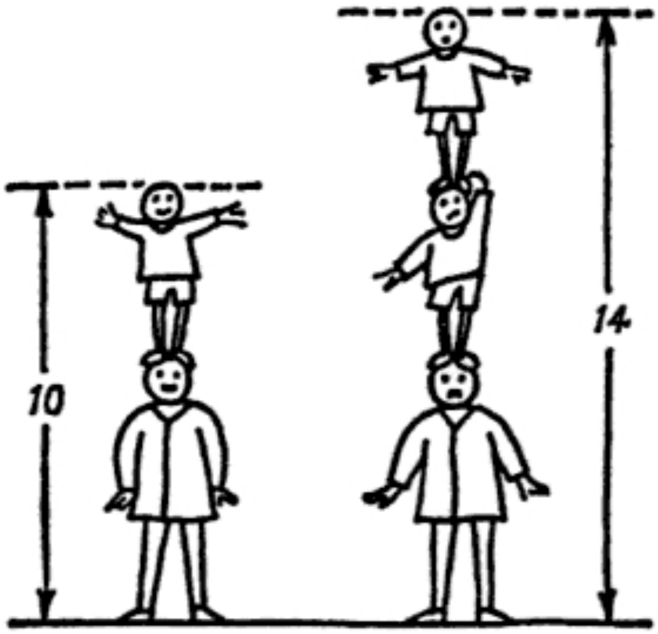


The Balancing Act

Solving Simultaneous Equations

2nd Year Ordinary Level Structured Problem Solving Research Lesson

Lesson Proposal developed by Mick Walsh, Brídín Roche (Teacher), Lisa O'Connor and Stephen McMahon engaging in *Lesson Study* with *Maths Development Team PDST*

Coláiste Eamonn Rís, Wexford

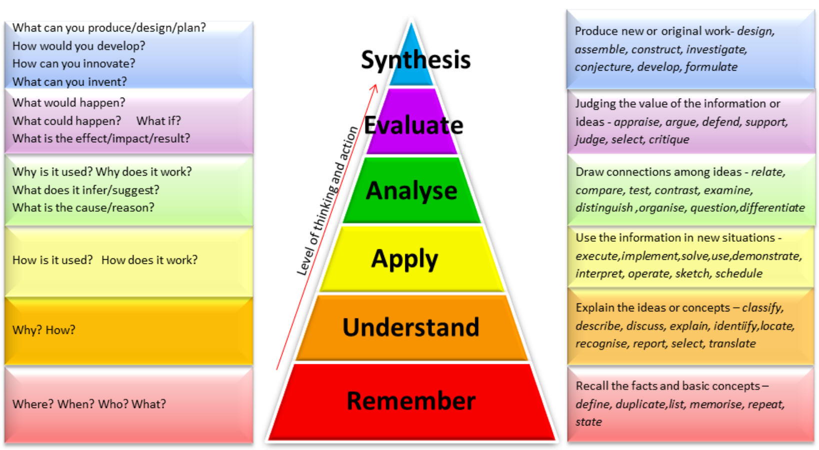
Table of Contents

Brief Description of the Lesson

In this lesson students will solve problems with two unknowns. Students will consider multiple approaches to solving simultaneous linear equations including: logical reasoning, trial and error, comparing tables, elimination with substitution, double substitution and by using a graph.

Research Theme

As a school we have decided to make Bloom's Taxonomy the focus of our school improvement plan. The diagram below shows the elements of this focus as they apply to secondary schools teaching/learning.



At Cólaiste Éamon Rís, we want our teachers to:

- use a range of questioning effectively for a variety of purposes including stimulating substantial student responses, facilitating deeper engagement with lesson content and extending learning beyond the lesson.
- value and engage in professional development and professional collaboration

At Cólaiste Éamon Rís, we want our students to:

- demonstrate knowledge, appropriate to their stage of development, of their own behavior as individuals and as members of a group.
- to apply this knowledge thoughtfully to manage situations and support their well-being.

As a mathematics department, we will actively support the achievement of these goals in the following ways:

- Subject knowledge: It is expected that all teachers in the department will have a deep understanding of the subject knowledge they are responsible for teaching. If there is a knowledge deficit, the onus is on the teacher to require the requisite skills through personal research. Lesson planning and structured problem opens the door to best practice through collaboration with colleagues in the maths department and through deep personal and collaborative reflection.

- Pedagogical content knowledge:_In terms of teaching and learning in our school we recognize that having the subject knowledge is only half the battle. Our teachers need to be current in terms of the most recent research in maths pedagogy. The whole school plan to use Bloom's Taxonomy and using a collaborative approach to subject planning and developments encourages our teachers to work together in implementing the most effective teaching/learning strategies. In the development of research lesson we had new teachers, teachers with circa five years, fifteen years and thirty years' experience coming together to engage in the process. This has to be good in terms of knowledge, pedagogy, proficiency in ICT and collegiality in the department.

- Management of teaching and learning: As Bloom's Taxonomy provides us with our focus on our questioning techniques we have agreed to use structured problem solving as a methodology to implement it and focus it in a meaningful way in our classrooms. It is hoped by this lesson planning team to introduce it to the maths department with the intention of using the approach in presenting new concept development and learning outcomes to students where possible.

Background & Rationale

Why we chose this topic

In the first meeting of the planning cycle we discussed what our lesson would be based on. Firstly we decided that we would present the research lesson to a mixed second year group. The reason for this was that the Principal allowed us some class time to engage in the planning process, and this was taken mostly from second years. We decided to use an ordinary level class as the subject of our research lesson.

In terms of subject matter for the lesson three issues immediately arose; (1) operations involving integers, (2) fractions including algebraic fractions, and (3) solving simultaneous equations. We decided on simultaneous equations, as ordinary level students rarely if ever acquire the concepts involved and are unable to even attempt the problems as they are presented in text books and in examinations. Furthermore we felt that the mathematics involved is extremely rich and allows us connect many areas of the curriculum, including patterns, co-ordinate geometry, algebraic procedures and development of problem solving skills. It also opens up many different ways to approach a problem, for example, numeric, graphical and algebraic methods.

Chief examiners reports on the SEC examinations over the years point to students having difficulty solving simultaneous equation, particularly when they are set in a practical context.

> *Most candidates also had difficulty solving equations (Question 11(a) and (c)). When required to solve a linear equation in part (a), many simply moved terms, unchanged, from one side of the equation to the other. Candidates also had difficulty solving the simultaneous equations in part (c). Many candidates used trial and improvement here rather than algebraic manipulation, but generally only substituted the correct values into one of the equations rather than into both of them. Those who did use algebraic manipulation often displayed a general idea of what to*

> *do, but were unable to carry out the procedures accurately and usually stopped when they had found a value for x only.* (Chief Examiners Report, Junior Certificate Examination 2015, p.23)

Our research findings

Following deep discussion by the lesson planning team, we decided on working with simultaneous equations. Most students are unaware of the fact that the equations represent geometrical entities and where lines intersect. We debated as to what approach we wanted to aim the students at. The team had teachers who came down in favour of using elimination and others who preferred some sort of equation/substitution approach. For junior certificate the textbooks currently favour an elimination approach to simultaneous equations and that is the method usually applied by our second year students. One teacher believes this is an incorrect approach to use for "weaker" students as they will never be required to solve equations with more than two unknowns so will never need to use an elimination approach. We decided to steer away from graphical solutions at this point but all agreed it should be followed in future lessons.

To address the concerns of all we decided to try to come up with some basic problem that would lend itself to be solved by both approaches. As discussed, only students with an excellent aptitude ever understand simultaneous equations so we focused of our research attention in coming up with very simple representations of a problem that all students could access.

Our research led us to discover the work of the mathematician and educator W. W. Sawyer who makes a compelling argument for the early introduction of simultaneous equation in his 1964 book, *Visions in Elementary Mathematics*:

> *It is quite possible to use simultaneous equations as an introduction to algebra. Within a single lesson, pupils who previously did not know what x meant can come not merely to see what simultaneous equations are, but to have some competence in solving them. No rules need to be learnt; the work proceeds on a basis of common sense. The problems the pupils solve in such a first lesson will not be of any practical value. They will be in the nature of puzzles.*

Sawyer gives an example of such a puzzle, stating that: "a man has two sons. The sons are twins; they are the same height. If we add the man's height to the height of one son, we get 10 feet. The total height of the man and the two sons is 14 feet. What are the heights of the man and his sons?" We decided to use and develop Sawyer's problem in a structured problem solving setting to address our lesson goals.

It is hoped that by presenting the problems in an accessible form such as this students will develop the required understanding of the concepts involved to have the confidence to successfully transfer the skills learned to solving more difficult problems. It is hoped that "weaker" students would be comfortable with a method and that "stronger" students would be confident using multiple approaches, as will be encountered in students later learning.

Relationship of the Unit to the Syllabus

Related prior learning Outcomes	Learning outcomes for this unit	Related later learning outcomes
<u>Overview third to sixth classes</u> <u>Skills development Skills</u> • Applying and problem-solving • Communicating and expressing • Integrating and connecting • Reasoning • Implementing The child should be enabled to • identify positive and negative numbers on the number line walk the number line to experience positive and negative numbers that arise in discussion and/or in context identify and mark positive and negative numbers on personal and class number lines • add simple positive and negative numbers on the number line add simple positive and negative numbers	<u>Elements from Junior Certificate syllabus.</u> **2.2 Co-Ordinate geometry** Co-ordinating the plane. Properties of lines and line segments including midpoint, slope, distance and the equation of a line. Intersection of lines. **3.4 Applied measure** Modelling real-world situations and solve a variety of problems. **4.1 Generating arithmetic expressions from repeating patterns** Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a	<u>Elements from Leaving Certificate syllabus.</u> **Foundation Level** **2.2 Co-ordinate geometry** Co-ordinating the plane. Linear relationships in real-life contexts and representing these relationships in tabular and graphical form. Comparing linear relationships in real-life contexts. The significance of the point of intersection of two linear relationships. **4.1 (a) Generating arithmetic expressions from repeating patterns** Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of

• know simple properties and rules about brackets and priority of operation, use the calculator in exercises to find missing numerals and missing operator. • explore the concept of a variable in the context of simple patterns, tables and simple formulae and substitute values for variables identify and discuss simple formulae. • translate word problems with a variable into number sentences. *(Primary School Mathematics Curriculum)*	correspondence from each input to each output. **4.2 Representing situations with tables, diagrams and graphs** Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next. **4.3 Finding formulae** Ways to express a general relationship arising from a pattern or context. **4.4 Examining algebraic relationships** Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships. **4.5 Relations without formulae** Using graphs to represent phenomena quantitatively. **4.7 Equations and inequalities** Selecting and using suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations and inequalities. They identify the necessary information, represent problems mathematically, making correct use of symbols, words, diagrams, tables and graphs. **5.2 Graphing functions** Interpreting and representing linear, quadratic and exponential functions in graphical form **Synthesis and problem-solving skills** − explore patterns and formulate conjectures − explain findings − justify conclusions − communicate mathematics verbally and in written form − apply their knowledge and skills to solve problems in familiar and unfamiliar contexts − analyze information presented verbally and translate it into mathematical form − devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. Elements from Junior Certificate syllabus.	inputs, a set of outputs and a correspondence from each input to each output. **4.1 (b) Representing situations with tables, diagrams and graphs** Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next. **4.1 (c) Finding formulae** Ways to express a general relationship arising from a pattern or context. **4.1 (d) Examining algebraic relationships** Features of a linear relationship and how these features appear in the different representations. Constant rate of change. **4.2 Solving equations** Solving linear equations set in context. **4.3 Inequalities** Solving linear inequalities set in context **Higher and ordinary levels.** **2.2 Co-ordinate geometry** – recognize the fact that the relationship $ax + by + c = 0$ is linear– solve problems involving a line and a circle **4.2 Solving equations** – select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form: • $f(x) = g(x)$, with $f(x) = ax+b$, $g(x) = cx+d$ • simultaneous linear equations with two unknowns and interpret the results • one linear equation and one equation of order 2 with two unknowns • simultaneous linear equations with three unknowns **5.1 Functions** – graph functions of the form – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions -use graphical methods to find approximate solutions to intersection of two functions

Goals of the Unit

- Students will understand that some types of problems that involve two unknowns give rise to linear algebraic expressions with two variables.
- Students will understand that the variables represent some concrete identity such as length, weight, number of, value of etc.
- Students will write material given as simple picture problems as simple equations.
- Students will write equations in pairs and solve them numerically by trial and error or some type of intuition.
- Students will solve pairs of similar equations by elimination.
- Students will solve pairs of similar equations by substitution.
- Students will be able to write equations from mathematical word problems.
- Select a preferred method to solve simultaneous equations.

Our Unit Plan

Lesson	Learning goal(s) and tasks
1	Revise algebra and solving linear equations using the stabilizer method (as discussed and written in our maths in the "Common Approaches" section of our Mathematics Department plan). This revision of first year work will allow us move straight into the research lesson "problem"
2 The Research Lesson	Introduce simultaneous equations in a simple problem solving context. Allowing multiple approaches including: • Trial and error (numerical) • Logical reasoning (problem solving approach) • By elimination with substitution (algebraic) • By double substitution (algebraic) • Using a table • By a pattern and graph drawing approach (graphically)
3	Practice skills relating to simultaneous equations
4	Practice skills relating to simultaneous equations
5	Solve problems in context

Goals of the Research Lesson

Mathematical Goals

Students will:

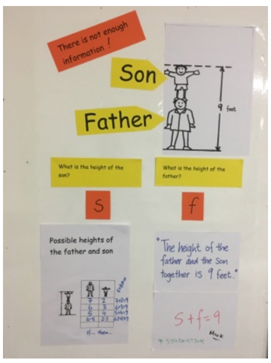
- Understand that some problems involve two unknowns. We require two equations to be able to solve a problem with two unknowns. We call these "simultaneous equations".
- If we knew the value of one of the unknowns then we could use this value to evaluate the other unknown.
- See that a diagram/picture can be useful when thinking about problems.
- Develop skills relating to forming mathematical equations and solving these using the "substitution method".
- Understand that these problems may possibly be solved by numeric, algebraic and graphical methods

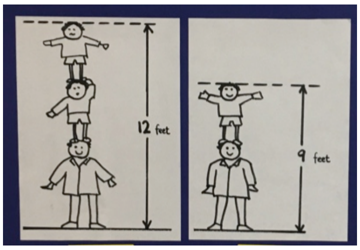
Key Skills and Statements of Learning

With the advent of the new Junior Cycle Specifications for Mathematics our maths department with the assistance of the school's senior management team has begun to investigate the development of the key skills and their integration into our maths planning and subject development. We have considered the relevance of each of the Junior Cycle *Statements of Learning* and *Key Skills* in the planning of this lesson. (Being Literate and Communicating) Through Ceardaíocht, students will have the opportunity and be encouraged to express their ideas clearly and accurately. (Being Numerate, Creative and Managing Information and Thinking): By engaging in suitable tasks, students will develop a positive attitude towards investigating, reasoning and problem solving. Students' will also explore options and alternatives as they actively participate in the construction of knowledge. They will be encouraged to think creatively and critically to select methodologies that suit their aptitude and approach to the subject matter Materials will be presented so that students will function at all levels of Bloom's Taxonomy. (Managing Myself and Working with Others) Students will learn with and from each other by discussing different problem solving approaches. They will have the opportunity to reflect on their own learning.

Flow of the Research Lesson:

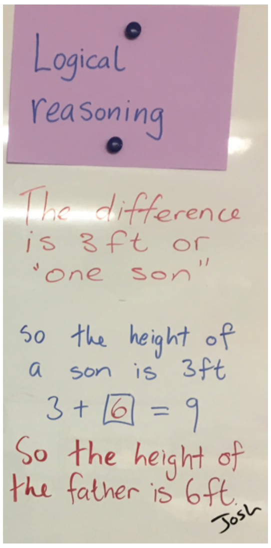
Note: As questioning is the focus of work in the whole school context questions will be asked to encompass all levels of Bloom's taxonomy. Questions of the following type will be asked: "What do you think"? (Ask student(s) other than the presenter). "Why is that"? "Did anyone else solve it the same way? Can you explain this method?" It is important that the whole class is engaged and that students understand they may be called on at any stage to reflect on what another student said.

Learning Activities and Expected Student Responses	Teacher Support	Assessment
Posing Task 1 (3 minutes) **"An American father has twin sons, both the same height. If we add the father's height to that of one of the sons we get 9 ft.** **What is the height of the father?** **What is the height of the son?"** What could the heights be do you think? Describe in words what you see. Make an equation to describe what you see. This task will be presented at the board. Students will not be required to work individually but rather think, share and discuss their ideas with important student ideas being recorded at the board.	The problem will be posed with the following picture.  Allow students the opportunity to dismantle the problem so that it suits their own individual attempt to attend to the task. Accept various answers. Write students suggestions on the board. (label each students work with their initials).	Do the students understand the task? Do students understand they are trying to find out or calculate two things they don't know at the same time in a single problem? Are students eager and engaged in the problem? Are the showing positive self-esteem in their attempts to solve the problem?
Presenting Student Responses: Task 1 (10 minutes) *"There is not enough information to know what the heights are." "There are many possible answers."* *"You can't work it out because you don't know what either of them is. If you knew one of them you could work out the other."* They heights could possibly be: "*3ft and 6ft", "2ft and 7ft", "3.5ft and 5.5ft"* … *"The height of the father and the son add to 9 feet."* If f is the height of the father in feet and s is the height of the son in feet, then: "$f + s = 9$" (Expect to see $x + y = 9$) 	Record the approaches on the board asking students to explain their thinking. In the event some of the students suffer a "maths block", offer them some advice and entry points into the problem. Some lower level Bloom's questions may help here. Emphasise the word height in their answers. This will help them to contextualize what they are about and hopefully learn the require concepts in a meaningful way. Acknowledge that there are a variety of possible answers to the problem. Record these in the form of a table. (This will support graphical solutions later).	Are students able to tackle the problem? Do student understand that they don't know either of the heights but do know the sum of the heights? Do they understand that the heights of the people may vary? Do the students recognize that they have developed an equation with two unknowns? Can the students give a correct solution? Can they explain their reasoning?

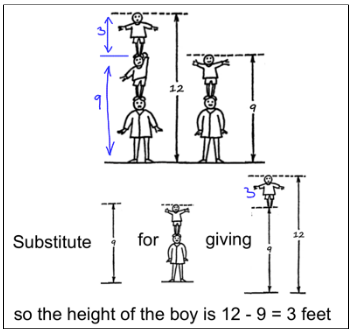
Learning Activities and Expected Student Responses	Teacher Support	Assessment
Posing Task 2 (3 minutes) **"An American father has twin sons. The sons are the same height. The height of a father and son is 9 feet. The total height of the father and his two sons is 12ft.** **In as many ways you can, work out:** **i. The height of a son?** **ii. The height of the father?"** This will be presented on the board. No picture will accompany it until the students have spent 5 minutes individually solving the problem. Students will be given worksheets with the "word" problem on it. Students will be asked to predict how many methods there are to solve the problem.	A little time will be spent clarifying the question.  The picture above will be added to the question during individual problem solving time.	Do the students understand the task?
Student individual work: Task 2 (12 minutes) Anticipated student responses: 1. Logical reasoning about the word problem 2. Thinking about the pictures 3. Trial and error (may include comparing tables) 4. Creating and solving equations	Can you think of another way? What can you deduce from the numbers? What can you see in the pictures? Why did you do that?	Are students engaged in the problem and trying different approaches with some success? Do the students recognise that they have developed two equations with two unknowns that are the same?

Presenting Students' Work: Task 2
(15 minutes)

1: Logical Reasoning

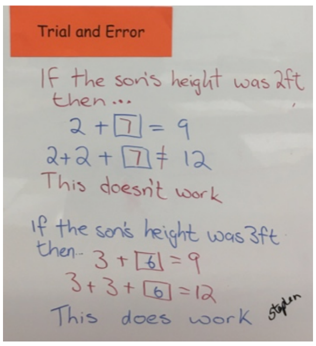


2: Thinking about the pictures



…and the height of the father is 6 feet

3: Trial and Error



3: Creating and solving equations using the substitution method

A $s + f = 9$

B $s + s + f = 12$

Substitute 9 for s + f

$-9 \quad | \quad s + (9) = 12 \quad | \quad -9$

$s = 3$

Substitute 3 for s

$-3 \quad | \quad (3) + f = 9 \quad | \quad -3$

$f = 6$

(Students may need support in developing this solution, if so the teacher will use the picture to progress this understanding.)

Which method do you think is best? Why? How can we check our answer?

Clarify the solution – heights in feet…

<table>
<tr><th>Learning Activities and Expected Student Responses</th><th>Teacher Support</th><th>Assessment</th></tr>
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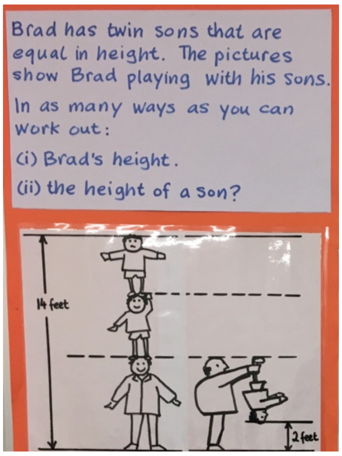
Posing Task 3

(2 minutes)



The task will be presented at the board and students will be given have worksheets with the problem.
</td><td>Clarify the task</td><td>Do the students understand the task?</td></tr>
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Student individual work: Task 3
(12 minutes)

Anticipated student responses:

1. Thinking about the pictures
2. Trial and error (may include comparing tables)
3. Creating and solving equations (elimination)
4. Creating and solving equations (substitution)
</td><td>Can you think of another way?
What can you deduce from the numbers?
What can you see in the pictures?
Why did you do that?</td><td>Are students engaged in the problem and trying different approaches with some success?
Do the students recognise that they have developed two equations with two unknowns that are the same?</td></tr>
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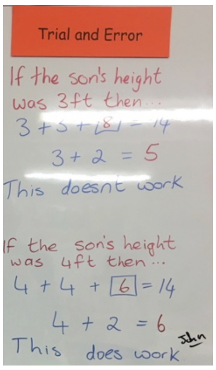
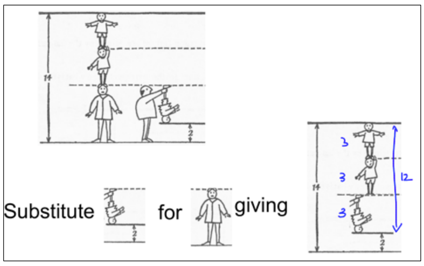
Presenting Students' Work: Task 3
(15 minutes)

<u>Trial and Error</u>
(This may or may not be shown.
A table might be used for this also.)



<u>Thinking about the Pictures</u>



<u>Create and solve equations using the substitution method</u>

Form equations and solve

C $2s + f = 14$

D $f = s + 2$

Substitute f for s + 2 into C

$2s + (s + 2) = 14$

$3s + 2 = 14$ (−2)

$3s = 12$ (÷3)

$s = 4$

Substitute 4 for s into D

$f = (4) + 2$

$f = 6$

Solutions: Height of father is 6 feet and height of the son is 4 feet.

Use picture to support this understanding
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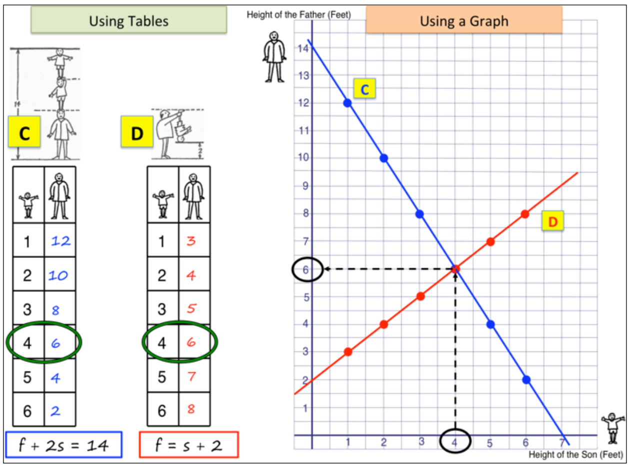
Ceardaíocht
(10 minutes)

Which method is the best? Why do you think that? What is the solution? How can we be sure that this is correct?

There is another way to think about this problem… Bring discussion to consider the linear patters, the tables comparing possible heights of the father and son and graphing these couples. A PowerPoint presentation will support this but it is the students that will complete the tables and graph on the board (subject to time).



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What did you learn today?

What are the important learning in this lesson do you think?

Which method do you prefer? Why?

Is there any of these methods that surprised you?

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Quality of engagement and responses.

What do students understand?

How do their responses relate to Bloom's Taxonomy?

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Summing Up and Reflection
(5 minutes)

Students complete: "Reflecting on my learning" worksheet" (See Appendix).

Have the students understand that problems like these are "simultaneous linear equations". Why are they called this do you think?

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What do students write about their learning and experience of the lesson?

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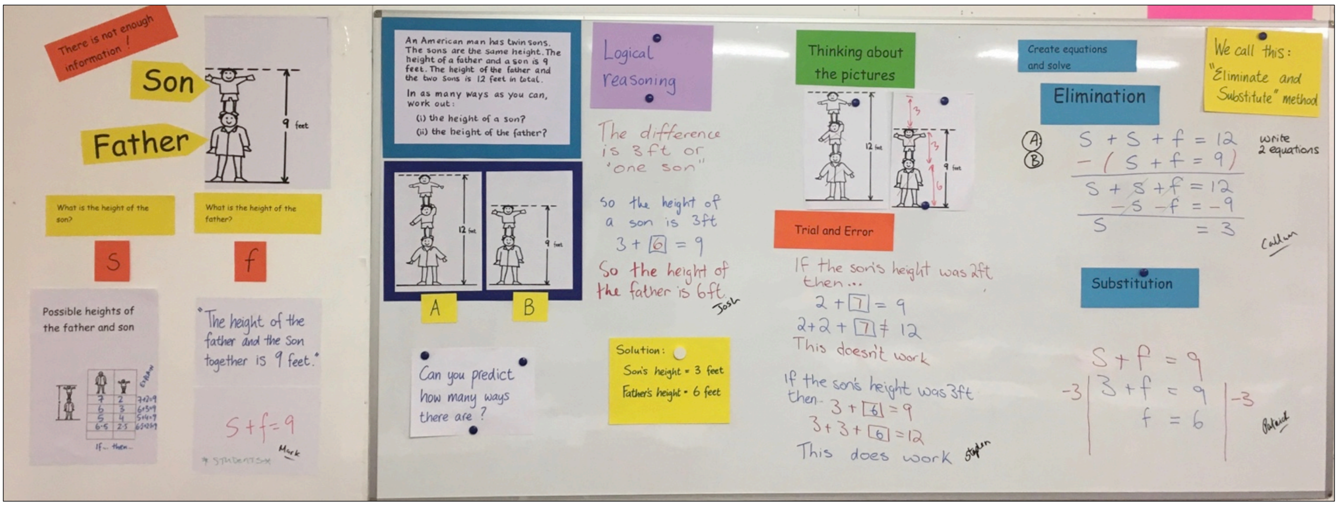
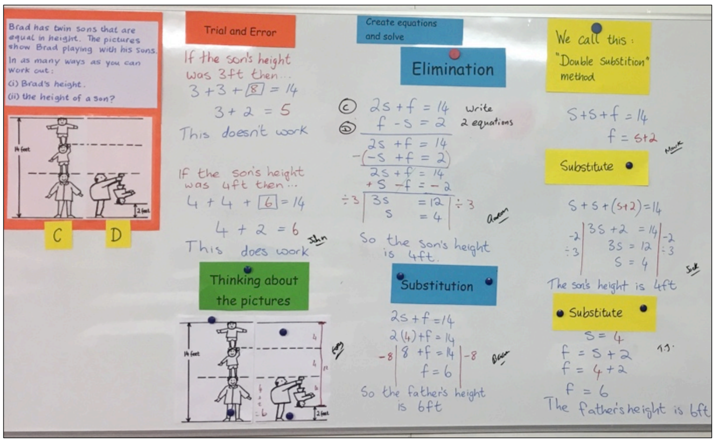
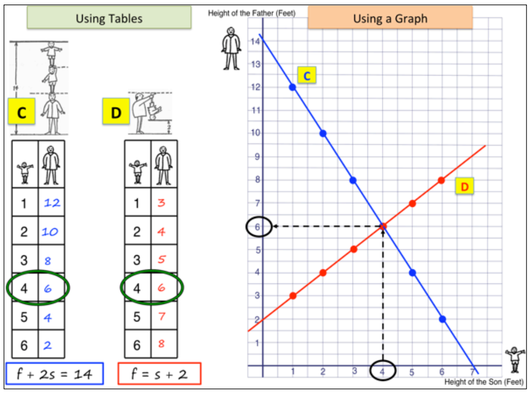
Our Board Plan



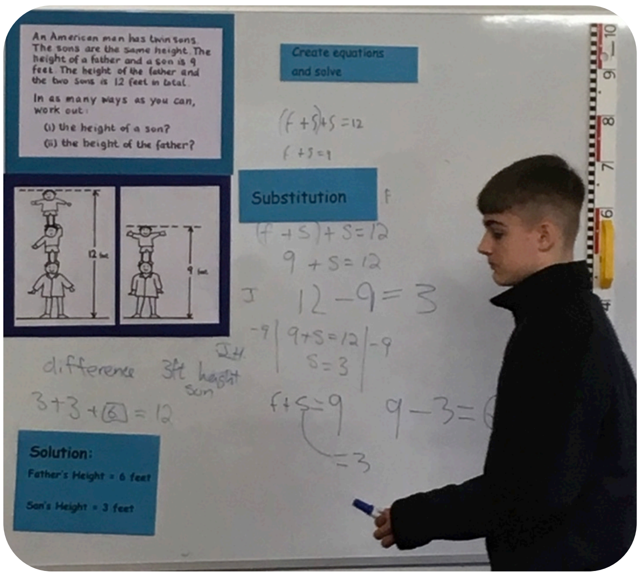




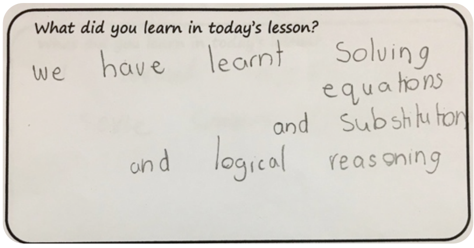
Reflecting on the Lesson

The lesson was held as planned on the last Tuesday of January. It was taught to Ms. Roche's second year ordinary level class over their timetabled double class. The students appeared to be engaged in the two classes at all times.

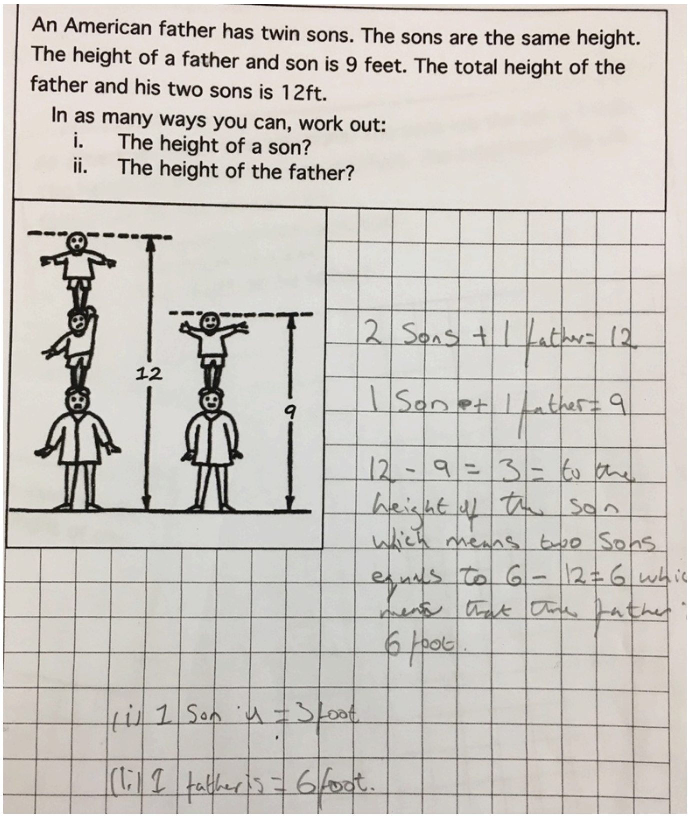
The students' responses after the class they were very positive. They enjoyed structured problem solving as a means for learning mathematics. The students' evaluations with one exception expressed a preference for using the structured problem solving approach.

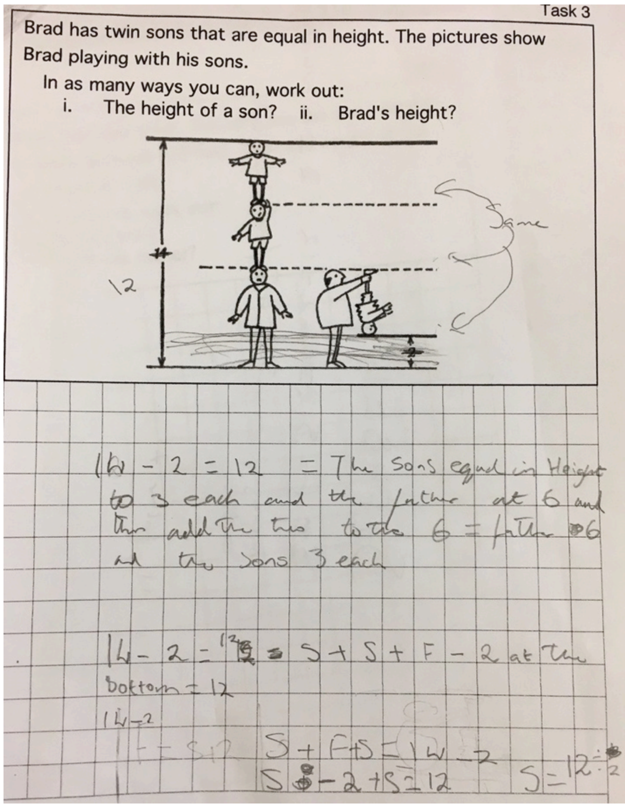


The teachers agreed that the mathematical aims of the class, all were met. Although the teacher if was felt correctly choose to leave the proposed graphical solution until the next class due to the pace of student progress during the class. The students have the necessary concepts to successfully solve problems with two unknowns. The final board was an excellent visual product that developed the learning and had all elements so it was easy to review what was done as required. A highlight of the class was when a student came up with the equation "F = B + 2" as this got to the heart of solving simultaneous equations by double substitution. It was noted that while all students were engaged during the lesson teachers were pleasantly surprised by the contribution of some of the "weaker" students. The lesson observation (see page 18) and post-lesson discussion also considered and acknowledge how all the levels of Bloom's Taxonomy were achieved and evident from students written work and oral presentations and discussion.



All the teachers involved were delighted with the finished product. The whole exercise was extremely valuable in terms of CPD. We all learned a lot about the process, engaged in active collaboration, and engaged in school planning at a very advanced level. One teacher commented that: "It was an excellent experience from start to finish and I will be teaching concepts using a structured problem solving approach in the future."





Appendices:

Worksheet 2.1

Task 2

An American father has twin sons. The sons are the same height. The height of a father and son is 9 feet. The total height of the father and his two sons is 12ft.

In as many ways you can, work out:

i. The height of a son?
ii. The height of the father?

Student:

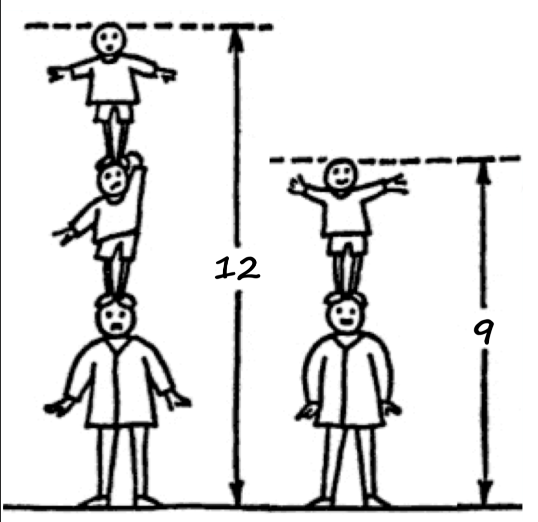
Worksheet 2.2

Task 2

An American father has twin sons. The sons are the same height. The height of a father and son is 9 feet. The total height of the father and his two sons is 12ft.

In as many ways you can, work out:

i. The height of a son?
ii. The height of the father?



Student:

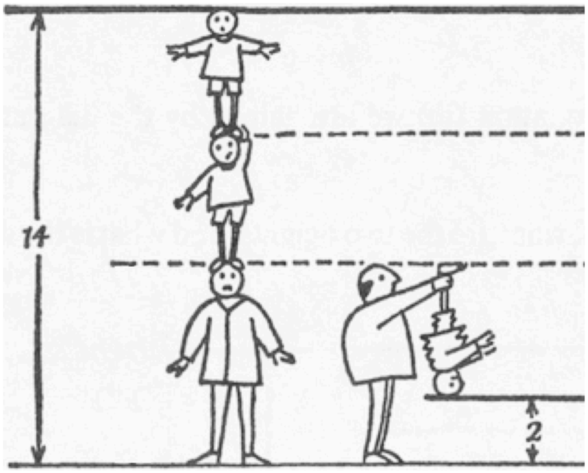
Worksheet 3.1

Task 3

Brad has twin sons that are equal in height. The pictures show Brad playing with his sons.

In as many ways you can, work out:

i. The height of a son? ii. Brad's height?

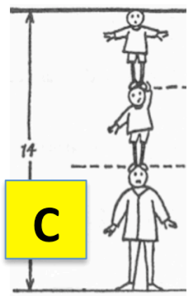


Student:

Worksheet 3.2

Using Tables

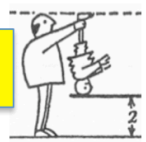
C



1	
2	
3	
4	
5	
6	

$f + 2s = 14$

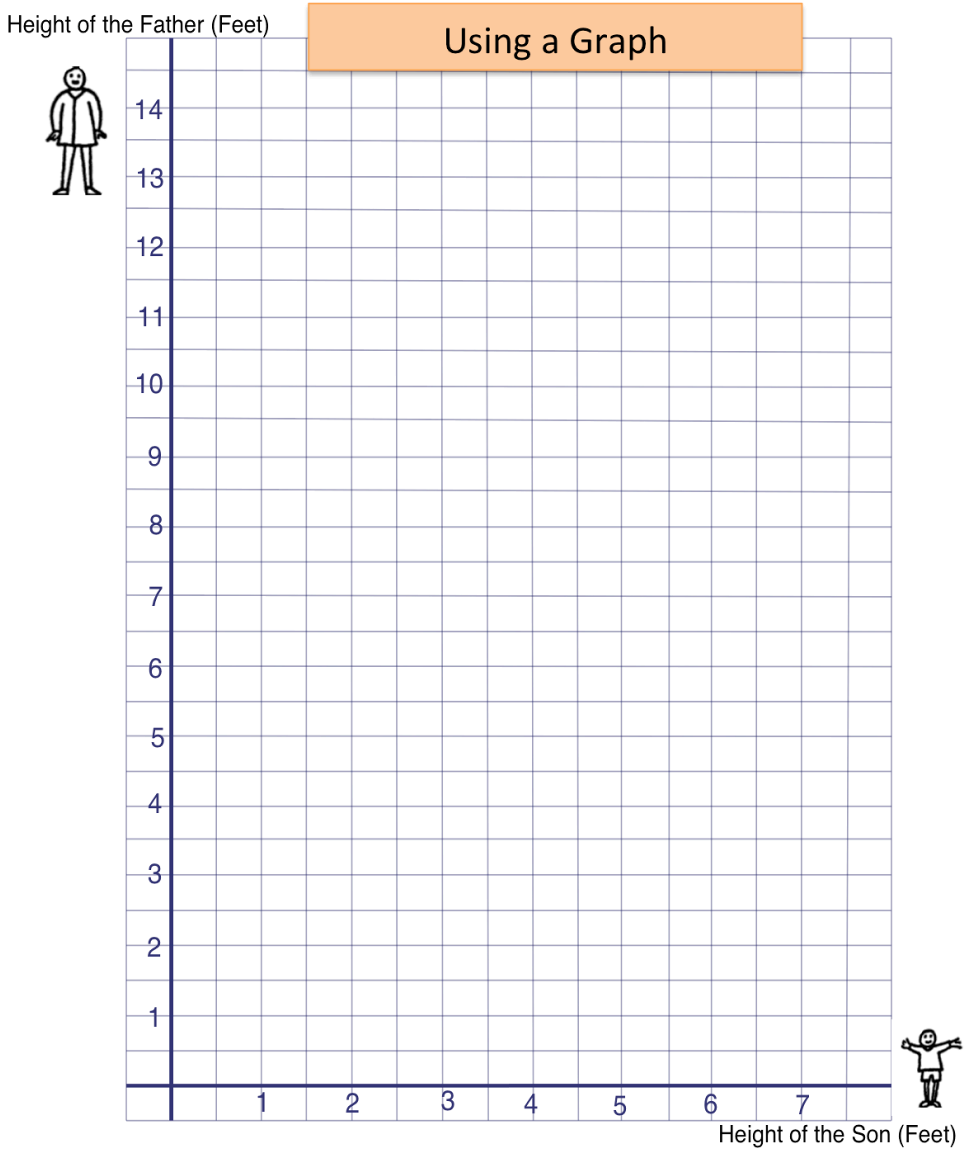
D



1	
2	
3	
4	
5	
6	

$f = s + 2$

Using a Graph



Student: ____________________

Student Reflection Sheet

Reflecting on my Learning

Student:

What did you learn in today's lesson?

Is there anything that you don't fully understand?

Did you enjoy todays lesson? Why or why not?

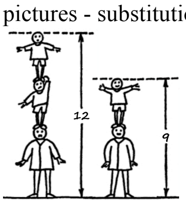
Lesson Study
Maths Development Team PDST

Observing Bloom's Taxonomy

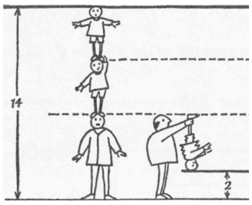
Blooms Taxonomy	Words/actions	Student names and Responses
Remember	**Action Verbs** Describe, Identify, Match, Recall, Recognise, Label, Quote, Name, Define, Recite **Potential student responses** Recall what an equation looks like ($6 + 3 = 9$) Unknowns/variables Point of intersection	
Understand	**Action verbs** Compare, contrast, illustrate, Relate, Summarise; Constructing meaning from oral/written/graphic material Explain answer **Potential student responses** Trial and error method estimating	
Apply	**Action verbs** Develop, Correlate, Calculate, Link, Manipulate Predict, Determine, Examine, Solve, Use **Potential student responses** Subtract fathers height from 9 to get son's height	
Analyse	**Action verbs** Develop, Correlate, Calculate, Link, Manipulate Predict, Determine, Examine, Solve, Use **Potential student responses** Subtract fathers height from 9 to get son's height	
Evaluate	**Action verbs** Interpret, Justify, Critique, Argue, Defend Judge, Compare, Reason **Potential student responses** Judging/ we don't have enough information If we knew one height we could work out other Thinking about the picture	
Synthesis (Create)	**Action verbs** Add to, Combine, Construct, Create, Elaborate, Modify, Produce, Simplify; Generate equation **Potential student responses** Creating equations Graph	

Teacher Observation Sheets

Teachers Observation Sheet – Task 1

	Response	Notes	Students to present
Task 2	Logical Reasoning		
	Thinking about the pictures - substitution 		
	Trial and error		
	Creating and solving equations		
	Misconceptions		
	Unexpected responses		

Teacher Observation Sheet – Task 3

	Response	Students – notes - misconceptions	Students to present
Task 3	Thinking about the pictures – substitution 		
	Trial and error		
	Creating and solving equations		
	Misconceptions		
	Unexpected responses		