# Lesson Research Proposal for Fifth Year Higher Level, Algebraic Fractions 

For the lesson on $1^{\text {st }}$ February 2018<br>At Mount Temple Comprehensive, Fifth Year Higher Level<br>Instructor: Rory Buckley<br>Lesson plan developed by: Rory Buckley, Robert Forest, Emer Brady

## 1. Title of the Lesson: The Grand Rational

## 2. Brief description of the lesson

Fifth year Higher Level students are introduced to an algebraic fraction. Using their mathematical skills they are asked to uncover as many properties of the fraction as they can. This begins with an attempt to understand the expression through their understanding of linear, quadratic, cubic and exponential graphs. Students are accustomed to using tables, graphs and algebra and can apply these methods to the algebraic fractions, exploring the similarities and differences they find.

## 3. Research Theme

As the Mount Temple Comprehensive mathematics department, we strive for students that enjoy their learning and are motivated to learn, and expect to achieve as learners.
The mathematics teachers intend to work together to devise learning opportunities for students across and beyond the curriculum.
The maths department supports this through developing teacher knowledge and competencies in the following areas:
a) Subject Knowledge: Teachers will work in collaboration with the department to identify any gaps their subject knowledge and to deepen their understanding of mathematical concepts to provide the best possible mathematical instruction. This will be achieved through professional discussion and collaboration and individual teacher reflection on practice with appropriate use of student input.
b) Pedagogical Knowledge: Teachers will work together to identify teaching strategies which will support student learning, using research based methods and accessing CPD where necessary
c) Classroom Management: The experience of the students in the classroom will be central to the planning for and implementation of teaching and learning in mathematics. The pedagogical approach used should encourage students to grow as learners, motivated and achieving their fullest potential. The use of active learning strategies will support this goal and teachers will share their successful pedagogical approaches.

## 4. Background \& Rationale

a) As a whole school community, the development of our students as independent learners was identified as a collective aspiration for our students. Although students display a positive attitude toward their learning, teachers agreed there was a need to reinforce and encourage independent learning throughout their time at Mount Temple. Our latest whole school inspection noted the expertise within the school and encouraged teachers to share their knowledge. With the teachers in the mathematics department holding a variety of skills from both within and beyond the teaching profession, there is the potential for teachers to work creatively together to build learning experiences which both fulfil and at times, exceed the curricular expectations. Following discussions among the teacher group the topic of algebraic fractions was highlighted as an area which challenged even the higher achieving students. The teachers wished to investigate methods of developing student understanding and move away from the purely mechanistic approaches employed by teachers at the present time. It was felt that the students often failed to associate algebraic fractions with normal fractions. The students had a limited understanding of the nature of
algebraic fractions and how they could approach an unknown function or expression and gain a deeper understanding of how the expression might be utilised in different circumstances.
b) In the initial stages of planning we attempted to find or develop a task which would relate algebraic fractions to real world situations. Following a number of attempts we decided the possible questions were too forced and did not easily apply themselves to algebraic fractions. At the same time we noted in the PISA results that Irish students struggled with pure maths and we decided to change our approach and look for an interesting and challenging task rather than focus on a real life application. We decided upon a difficult rational expression which as a function contained two roots and two zeroes. This expression would allow students to apply algebraic methods. They could then consider the expression as a function and compare their knowledge of rational functions to lines and quadratics. Students should be able to plot the function using the table function on their calculator. They should notice an error on their calculator and attempt to explain why this happens and what it means for their graphs. They can locate roots both through their graph and algebraically. As they use $f(x)=0$ to find the roots, important ideas around equality, dividing by zero and the difference between the solution and the original function may be highlighted and become more apparent to the students. Simpler expressions were considered but these had a single root and would not have offered the level of challenge needed to engage students and deeply challenge their understanding. All of the mathematical knowledge, content and understanding needed to complete the task are within the skill set of a higher level student and the task would encourage students to use their knowledge in a novel environment, with creativity and deepen their understanding.

## 5. Relationship of the Unit to the Syllabus

| Related prior learning Outcomes | Learning outcomes for this unit | Related later learning outcomes |
| :---: | :---: | :---: |
| U3. Students should be able to recognise that equality is a relationship in which two mathematical expressions have the same value. <br> U13. Students should be able to communicate mathematics effectively: justify their reasoning, interpret their results, explain their conclusions, and use the language and notation of mathematics to express mathematical ideas precisely. <br> AF3. Apply the properties of arithmetic operations and factorization to generate equivalent expressions so that they can develop and use appropriate strategies to add, subtract and simplify | To connect graphical and symbolic representations of algebraic concepts <br> To use appropriate graphing techniques (graphing calculators, computer software) throughout the strand activities <br> Students should be able to add and subtract expressions of the form $\frac{a}{b x+c} \pm \frac{p}{q x+r}$, <br> where $a, b, c, p, q, r \in Z$ <br> Students should be able to select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form $f(x)=g(x)$ | Students should be able to apply their learning novel situations. <br> Students should be able to apply mathematical processes such as integration and differentiation to algebraic fractions and understand the limitations of such processes, with regard to discontinuous functions and apply concepts such as injective, surjective and bijective and understand their relevance. |


| Expressions of the form <br> $\frac{a}{b x+c}$, where $a, b, c \in Z$ | with $f(x)=\frac{a}{b x+c} \pm \frac{p}{q x+r} ;$ |
| :--- | :---: |
| AF.4 to select and use suitable <br> strategies (graphic, numeric, | $g(x)=\frac{e}{f}$ |
| where $a, c, d, r$ it |  | algebraic, mental) for finding solutions to quadratic equations in one variable with coefficients in $\mathbf{Q}$ and solutions in $\mathbf{Z}$ or in $\mathbf{Q}$.

Linear inequalities in one variable of the form $g(x)<k$ and graph the solution sets on the number line,
$x \in N, Z$ and $R$
AF. 5 form quadratic equations given integer roots

AF. 7 investigate the concept of a function so that they can Demonstrate understanding of a function
Represent and interpret functions in different waysgraphically (for $\mathrm{x} \in$ $N, Z$ and $R$ ( continuous functions only), as appropriate), diagrammatically, in words, algebraically- using the language and notation of functions (domain, range, codomain, $f(x)=, f: x \rightarrow, y=$ ) Use graphical methods to find and interpret approximate solutions of equations such as $f(x)=g(x)$ and approximate solution sets of inequalities such as $\mathrm{f}(\mathrm{x})<\mathrm{g}(\mathrm{x})$.
Make connections between the shape of a graph and the story of a phenomenon, including identifying and interpreting maximum and minimum points.

$$
\begin{gathered}
\text { with } f(x)=\frac{a}{b x+c} \pm \frac{p}{q x+r} \\
g(x)=\frac{e}{f}
\end{gathered}
$$

where $a, b, c, d, r, e, f, p, q \in Z$

Students should be able to select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form:
$f(x)=\underset{a}{g}(x)$
with $f(x)=\frac{p}{b x+c} \pm \frac{p}{q x+r} ;$

$$
g(x)=\mathrm{k}
$$

where $a, b, c, d, e, k, r, p, q \in Z$
Students should be able to select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form:

$$
\begin{gathered}
g(x) \leq k, \quad g(x) \geq k \\
g(x)<k, \quad g(x)>k \\
\text { with } g(x)=a x^{2}+b x+c \\
\text { or } g(x)=\frac{a x+b}{c x+d}
\end{gathered}
$$

$$
\text { and } a, b, c, d, k \in Q, x \in R
$$

Students should be able to use notation $|x|$
be able to select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form:
$|x-a|<b,|x-a|>b$
And combinations of these, with $a, b \in Q, x \in R$

## 6. Goals of the Unit

The academic goals of this unit are that students use their understanding of fractions, algebraic fractions, quadratic expressions, equations and functions to develop and construct learning which leads them to a deeper understanding of algebraic fractions with a particular emphasis on representation and investigation of algebraic fractions as functions. Leading into equality and inequalities and exploring solutions as different from the expressions themselves. Continuing on students will then move to absolute value; notation, as functions and finding solutions, and solution sets for inequalities. While attempting to achieve our academic goals teachers will remain conscious of the learning experience of the students particularly that they are challenged sufficiently to stimulate interest in mathematics and create a deep understanding which allows them to utilize their procedural knowledge to approach unusual problems in both mathematical applications and in pure mathematics. Students will be encouraged to use appropriate mathematical language to discuss their ideas in small groups to construct understanding. Students will share with the class group as a whole, investigate their thinking and develop their mathematical communication skills.

## 7. Unit Plan

| Lesson | Learning goal(s) and tasks |
| :---: | :--- |
| 1 <br> The Research <br> Lesson | The research lesson will introduce algebraic fractions as an expression and <br> challenge students to consider the different representations available to <br> investigate algebraic fractions. |
| 2 | Practice of plotting algebraic fractions and identifying roots and invalid solutions <br> (40 mins) |
| 3 | Working on from the previous questions to consider inequalities (40 mins x2) |
| 4 | Absolute value expressions, using real life problem where negative solutions <br> would not be appropriate to explore these types of functions (40 mins) |
| 5 | Absolute values and inequalities (40mins x2) |

## 8. Goals of the Research Lesson

The mathematical goal of this research lesson is that students would utilize previous knowledge of fractions, algebraic fractions and quadratic equations to investigate rational expressions. Students will be encouraged to examine algebraic fractions as rational functions through mental representations, tables, graphing and algebra. Particular attention will be paid to features of functions and how these appear in graphical representations such as roots and asymptotes. The importance of equality, how the solutions appear as a quadratics which allow us to find roots but are very different from the original expression and it's representation as a function. The undefined values for x which result in a zero denominator and why this is a problem for maths. The structure of the lesson will provide opportunities for constructivist learning and for students to express mathematical ideas. The students will have experience of using and developing mathematics and mathematical ideas. They will work with other and communicate their ideas (KS4, 6 and 7). The students will describe, illustrate, interpret and explain patterns and relationships(SL16).

## 9. Flow of the Research Lesson:

| Steps, Learning Activities Teacher's Questions and Expected Student Reactions | Teacher Support | Assessment |
| :---: | :---: | :---: |
| Introduction and Posing the Task <br> What do you know about this expression? $\frac{2 x+4}{x-1}+\frac{2 x+1}{1-3 x}$ <br> There is no additional wording used in posing this problem such as placing it equal to a number, 0 or $\mathrm{f}(\mathrm{x})$ <br> Do we know anything about this expression? How would we find out? | We would have a lot more to say about $x^{2}+4 x+4$ <br> What could you do with this expression and can you attempt something similar with the expression you have been given | The teachers are hoping to make a list of keywords and phrases. These would include expression, equation, pattern, fraction, algebra, unknowns, function, tables graphs, solutions, linear, quadratic, cubic, exponential. |
| Using the list of words explore the expression, finding out as much as you can about how the expression might behave. <br> Student would have completed functions before the research lesson and could be prompted by asking what do they know about a quadratic where they would be able to identify intercepts, roots, shape and finding solutions for different y values, plot, sketch, put in a table, inspect for rates of change (common difference), minimum/maximum points, etc. | Written onto the board |  |
| Student Individual Work <br> Response 1 (Algebraically changing to a single fraction) <br> What does thir tell us? | Response 1 <br> This method although algebraically correct offers little in terms of what the expression can represent. Students understanding of whether these two versions represent the same expression should be checked and how would we know? Can we plot this is it the same? |  |




|   | Using algebraic methods the second root can be found. In order to achieve this, students must understand the value of the denominator is not important and that we can solve by putting the numerator equal to zero. Students solve a quadratic equation to find the roots. However, this quadratic is clearly not our function but rather has roots in common with our function. <br> The graph cannot cross the line $x=1 / 3$ and $x=1$. Why is this the case? For any fraction a zero denominator is undefined. <br> We say this function is not continuous at these two values of $x$. |  |
| :---: | :---: | :---: |
| Summing up \& Reflection | When faced with an unfamiliar expression, there are a number of ways to attempt to gain a deeper understanding. <br> - Simplification using algebra <br> - Treating it an expression as a function and plotting <br> - Finding roots <br> - Finding zeroes <br> (if real roots and zeroes exist) <br> When we place an expression equal to zero we are finding the roots and the quadratic we use shares the same roots but is not the same as the rational function. |  |

In mathematics,
a rational function is
any function which can
be defined by a rational
fraction, i.e. an algebraic
fraction such that both
the numerator and the
denominator are
polynomials. The
coefficients of the
polynomials need not
be rational numbers;
they may be taken in any
field K.
We can use similar
techniques to investigate
an unfamiliar function
such as tables and
graphs. We can find $y$ -
intercepts and roots if
they exist.
For some functions
where a denominator of
zero is possible such
outcomes are undefined
and represent a break in
the graph of the
function, i.e. the
function cannot pass
these points and is
discontinuous.




| $\frac{2 \mathrm{x}+4}{\mathrm{x}-1}+\frac{2 \mathrm{x}+1}{1-3 \mathrm{x}}$ |
| :--- |
| What can you tell me about |
| this expression? | Are we more familiar with | expressions like $x^{2}+2 x+1$ ? |
| :--- |
| We have a vocabulary to |
| describe this expression and |
| methods to explore it. Can we |
| apply tables, graphs and |
| algebraic manipulation to this |
| expression to try to |
| understand it better? |

[^0]
## 11. Evaluation

The theme of this lesson study was to encourage students to grow as learners, to be motivated and achieve their fullest potential. We wanted to provide lessons which supported this aspiration and to share best practice across the maths department. In the course of our initial discussions we agreed there was a need to reinforce and encourage independent learning. During the lesson the students worked independently for an extended period of time on an unfamiliar problem which fulfilled our lesson theme. The teachers wished to investigate methods of developing student understanding of algebraic fractions and move away from the purely mechanistic approaches employed by teachers. Instead of just simplifying an expression and solving for a single value, the students could see how the value of the expression changed over a range of values and see the undefined values. The lesson allowed the students to see algebraic fractions as numbers which was a key concern of the teachers. Several misunderstandings were challenged and the understanding of fractions was deepened. Although most students attempted to simplify the expression, with a little encouragement the students started to investigate the expression through functions, plotting the function and finding roots and zeroes. A number of students immediately associated the function with a quadratic and solved for $\mathrm{f}(\mathrm{x})=0$ finding the roots through the quadratic and when asked to try to plot the function plotted the quadratic. This highlights a boarder misinterpretation and confusion between an expression or function and the solution (see examples of students work below).



## 12. Reflection

The teachers expected to see students attempt to simplify the expression and all the students set about simplifying immediately. The teacher foresaw the students putting the simplified expression equal to zero. The teachers' suspicion that the students would then see the expression as quadratic was confirmed with some students continuing to plot the quadratic rather than plotting the original expression as a function. While plotting the rational function the students found the error on their calculator. Some students changed the step on their calculator to investigate region around the error and to look for the roots. Using algebra to find the roots encouraged the students to consider how zero is written as a fraction and how the denominator can be ignored in solving equations.

A number of misconceptions or ideas which are not fully investigated in the course of their study were highlighted in this lesson study. Students have powerful tools to deal with unusual expressions and functions and are capable of applying these in novel situations. However, at times their understanding is limited, the lack of clarity between solutions and expressions was very clear, particularly when the students plotted the quadratic expression which provides them with the roots. We felt we took for granted the students understanding of dividing and multiplying by zero and we were surprised by the number of students who did not have a good understanding of the implications of zeroes in fractions. We felt the logical extension of the task would be to look at inequalities which pose similar issues for students and would strengthen their understanding.

The post lesson discussion congratulated the teacher on the delivery of the lesson. The structure of the lesson worked well and the timings were appropriate, although the students are accustomed to 40 minute classes and the energy levels definitely dipped beyond the usual class length. The students were engaged and challenged. They displayed good skills in algebra and graphing. The misconceptions and stumbling blocks foreseen in the lesson planning were evident and for the most part the lesson progressed as expected. The student reflections were then analysed. These were generally positive with students enjoying using show-me boards. However, students worried that their work was not permanently recorded in their copies. The students liked working on an extended question with time to investigate and reflect on their work. One student would have preferred to do lots of examples rather than just one and another student wanted to only use algebra. For the teachers, there was a realisation that the methods used for higher level maths still revert to very traditional teaching far too often and that students both enjoy and see value in greater exploration in their mathematics classrooms.

During the lesson the idea of inequality was introduced and certainly could be used as an extension question. This approach would solidify the ideas already covered in the lesson and also provide a springboard into absolute value inequalities where a quadratic is used to find the range of values and the question could be revisited in calculus.

Separating out the two parts of the fraction and plotting separately is possible to assist students in understanding how adding functions works and this leads directly into calculus where

$$
\frac{d}{d x}[f(x)+g(x)]=\frac{d}{d x} f(x)+\frac{d y}{d x} g(x)
$$

A simplified version as shown below could be used with junior or TY students. Our experience with the higher level students suggests this type of expression does not offer enough challenge and is not as rich in learning opportunities as the compound version.

Function

- $f(x)=-\frac{(2 x-4)}{x+1}$
- $h(x)=(2 x-4)(x+1)$

Line $g: x=-1$
Point

- $A=(2,0)$



[^0]:    Student Response 1
    

    Algebraic manipulation offers little insight into the expression only producing an alternative representation.

