Lesson Research Proposal for 6th Year Higher Level

Topic: Proof of Theorem 11 on transversals

For the lesson on 23 January 2018
At Coláiste Iognáid, Ms Margaret Gill's 6th Year HL class
Instructor: Margaret Gill
Lesson plan developed by: Margaret Gill, Órlagh Nic Eoin, Claire Lohan, Bernie Brennan

1. **Title of the Lesson: *Transversals are copycats***

2. **Brief description of the lesson**
Given a diagram with three parallel lines and three transversals, students will use their knowledge of geometry to identify existing relationships between pairs of angles, pairs of line segments and pairs of triangles, which will in turn give rise to a new relationship: *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal intercepts on any other transversal line* (Theorem 11).

3. **Research Theme**

Referring to *Looking at our School 2016 – A Quality Framework for Post-Primary Schools* we identified the following goals as a priority for the improvement of teaching and learning:

a) We want our students to grow as learners through respectful interactions and experiences that are challenging and supportive.
b) As teachers, we value and engage in professional development and professional collaboration.

As maths teachers we will address these goals by:

a) - designing/selecting suitable challenging problems which will encourage our students to think insightfully and creatively
   - providing our students with opportunities to come up with their own approaches to solving a problem
   - developing their confidence, competence and communication skills through expressing their ideas to their peers in their own words which enables the comprehension of all students in the class.
b) engaging in the collaborative Lesson Study process with a view to:
   - improving mathematics teaching and learning and students' experiences in the mathematics classroom
   - increasing the emphasis on the links between the strands of the mathematics syllabi
   - developing a consistent department-wide approach to the teaching of mathematics in our school.

4. **Background & Rationale**

a) Why we chose the topic

Our students generally enjoy manipulating, constructing and calculating the measurements of geometrical figures, however they seem to find it hard to engage in more rigorous deductive reasoning that is required in geometrical proofs. Students struggle when presented with abstract concepts.

In this project, we are going to focus on ratio and similarity theorems (Theorems 11, 12 and 13) and their proofs. Our lesson is aimed at 6th year students (Higher level). We usually teach the proofs by explaining to our students the steps involved. We would like to improve this approach by letting our students identify the relationships using what they have learned before and guide them in such a way that they would gain the understanding of the proofs through their own efforts. As teachers, we recognise the importance of prior learning in that it provides the scaffolding when learning something new and moving through mathematics syllabi.

Furthermore, through our discussion on mathematics learning we have come to realise that over dependency on formulae is problematic and that students must be given a chance to engage in non-procedural, active-learning. Synthetic geometry does not rely on procedures/formulae and therefore may not appeal to students who would normally reach for a formula straight away. We have concluded that we must provide our students with opportunities to develop the ability to examine the shapes logically, and hence develop their mathematical thinking and personal investigative skills that transfer to other areas of mathematics.

b) Our research findings

Geometry for Post-primary School Mathematics[1] that sets out the course in geometry states that:

- all of the geometrical results on the course would first be encountered by students through investigation and the ideas involved in a mathematical proof can be developed even at this investigative stage,
- when students engage in activities that lead to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday,
- a formal proof only allows us to progress logically from existing results to new ones.

This resonates with the Van Hiele theory that describes how students learn geometry[2]. According to this theory, students progress through various levels of thinking and cannot function at any particular level unless they are competent at all previous levels:

- Level 0 – students recognise the geometrical shapes
- Level 1 - students think about what constitutes a given shape (properties of shapes)
- Level 2 – students discover new properties by simple deduction and explain logically without having to measure everything
- Level 3 – students learn how to write a formal proof building on their prior knowledge.

The ability to construct proofs is an important skill for all mathematicians. Schoenfield (1994) suggests that proofs should not be viewed as separate entities in the curricula as a proof is *"an essential component of, doing, recording and communicating mathematics"*. In his article *"Proof as a Tool for Learning Mathematics"* Knuth (2002) suggests that as mathematicians first, maths teachers tend to recognise, and perhaps over-emphasise the primary role of the proof in mathematics, namely to establish the truth of the result. This results in a lack of emphasis being placed on the proof as an educational tool and no recognition of *"its role in fostering understanding of the underlying mathematics"*. Oftentimes, proving is perceived by students as *"a formal and often meaningless exercise to be done for the teacher"* (Alibert, 1988).

Knuth (2002) suggests that teachers create learning opportunities for students during which they encounter different types of proofs. This should result in some students developing a deeper understanding of both the proof, but also, the mathematics which underpins it. He suggests that students present their arguments to the class which gives opportunity to:

- discuss the explanatory quality of various arguments presented by their classmates,
- encourage students to seek further insight as to why certain statements are true,
- help demonstrate relationships amongst areas of mathematics that the students hadn't previously spotted.

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[1] Section B of the JC and LC Mathematics syllabi, Subsection 8.2 From Discovery to Proof
[2] See Appendix A in all Teacher Handbooks developed by the Maths Development Team

Since the investigation and use of Theorems 11, 12 and 13 are part of the JC HL geometry course, we expect that our LC HL students are familiar with these results and can apply them to solve problems. We plan to enable our students to progress from Level 2 to Level 3. Having seen in the introduction to Lesson Study how students can be led to a proof of Theorem 19 regarding angles in a circle, we plan to design a task which is closely related to the proof of Theorem 11 (*If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal*.) and requires students to use their prior knowledge and identify the relationships that are crucial to the proof.

5. Relationship of the Unit to the Syllabus

Related prior learning outcomes[3]	Learning outcomes for this unit[4]	Related later learning outcomes
From First Year through to Third Year, students learn about the concepts of congruent triangles, similar triangles, parallel lines, parallelogram, supplementary angles, vertically opposite angles, transversal line, alternate angles, corresponding angles, ratio, opposite sides or angles of a quadrilateral. Students also learn about axioms, including the axiom of parallels and the axiom on congruent triangles. Students study constructions and are exposed to some formal proofs. Students are expected to be able to (Strand 2, 2.1 Synthetic geometry): – *recall the axioms and use them as an aid to solve problems* – *understand and use terms: **theorem, proof, axiom, corollary, converse and implies (HL)*** – *apply knowledge of all theorems (including **Theorems 11, 12** at HL and 13 at OL), converses and corollaries to solve problems* – *prove the theorems **4,6,9,14 & 19 (HL)*** – *construct a line parallel to given line, through given point.*	**At Leaving Cert OL and HL,** knowledge of the axioms, concepts, theorems and corollaries from the corresponding syllabus level at JC is assumed. Learners should become familiar with the formal proofs of the specified theorems (proofs of Theorems 11, 12 and 13 are examinable at HL). Students are expected to (Strand 2, 2.1 Synthetic geometry): – *investigate theorems 11, 12, 13 and use them to solve problems* – ***prove theorems 11, 12, 13 concerning ratios (HL).*** We hope that students will: – engage in active learning when approaching theorems – appreciate prior learning in theorems and use it as a scaffold to understand problems – be able to add constructions to diagrams in order to produce a solution.	At each syllabus level students should be able to apply their knowledge and skills to solve problems which are unfamiliar. Learners should come to appreciate that certain features of diagrams are independent of the particular examples chosen, and that these constant features or results can be established in a formal manner through logical proof (p.23). The proofs of Theorems 11, 12, 13 lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry (work with trigonometric ratios).

6. Goals of the Unit

a) Cognitive Goals

[3] Department of Education and Skills, "Junior Certificate Mathematics Syllabus" for examination from 2016
[4] Department of Education and Skills, "Leaving Certificate Mathematics Syllabus" for examination from 2015

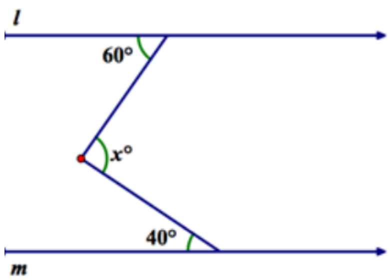
- Students will understand that sometimes drawing an additional line to a diagram helps to solve a problem (students will derive a proof for the sum of the three angles in a triangle and find the value of a missing angle in a given diagram).
- Through investigation of a suitable problem, students will review their prior learning from Junior Certificate, concerning Theorem 11.
- Through investigation, discussion and deduction students will arrive at and understand the proof.
- Students will apply Theorem 11 in familiar and unfamiliar contexts.
- Through investigation, discussion and deduction students will arrive at and understand the proof of Theorem 12.
- Students will apply Theorem 12 in familiar and unfamiliar contexts.
- Through investigation of a suitable problem, students will review their prior learning from Junior Certificate and Theorems 11 and 12, concerning Theorem 13.
- Through investigation, discussion and deduction students will arrive at and understand the proof.
- Students will apply Theorem 13 in familiar and unfamiliar contexts.

b) Emotional Goals

Students will:

- grow in confidence relating to their knowledge and the application thereof;
- appreciate the importance of geometry and its real-life applications;
- feel a sense of achievement and ownership of the concepts relating to the unit;
- develop a growth-mindset and apply a *can-do* attitude towards problem solving;
- work collaboratively, and thus appreciate the value of engaging in teamwork;
- improve their presentation and communication skills.

7. Unit Plan

Lesson	Learning goal(s) and tasks
1	*Review of prior learning at Junior Cert* re: vertically opposite, alternate and corresponding angles and triangles. Students will understand that drawing an additional line to a diagram may help to solve a problem. Problem 1: Prove that the three angles in any triangle add to 180°. Problem 2: Find the value of x in the diagram below in as many different ways as you can. Lines l and m are parallel. 
2	*Review of prior learning at Junior Cert* re: congruency. Students apply their knowledge of congruent triangles to solve problems, including the properties of parallelograms.
3 The Research Lesson	*Structured problem-solving lesson* Students investigate and derive relationships that lead to a proof and formal statement of Theorem 11: *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

4	*Follow up lesson regarding the proof of Theorem 11.* Students collate and structure the material from the previous lesson to derive a proof.
5	Application of Theorem 11.
6	Students investigate Theorem 12: *If a line l is parallel to side [BC] of ∆ABC and divides [AB] in the ratio s:t, it also divides [AC] in the ratio s:t.*
7	Using what they have learned in previous lessons in the unit, students prove Theorem 12.
8	Application of Theorem 12.
9	Using what they have learned in previous lessons in the unit, students prove Theorem 13: *If two triangles, ∆ABC and ∆A'B'C' are similar, then their sides are proportional, in order.*
10	Application of Theorem 13.

8. Goals of the Research Lesson

a) Mathematical Goals

Students will:

- identify the relationships between angles and line segments in the diagram leading to an understanding of Theorem 11 and its proof;
- generate a formal statement that *if three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal*;
- develop their confidence in deductive abilities;
- foster a positive attitude and a growth mindset towards theorems and proofs.

b) Key Skills and Statements of Learning

This lesson will promote Key Skills in the following ways:

1. Being Literate: Students will expand their mathematical vocabulary and express ideas clearly and accurately.
2. Managing Myself: Students will be able to reflect on their own learning.
3. Staying Well: Students will be more confident in their approach to Geometry.
4. Managing information and thinking: Students will be encouraged to gather and organise information in order to benefit their understanding.
5. Being Numerate: Students will see trends and relationships, which may not have been as obvious.
6. Being Creative: Students will implement ideas and take action in constructively solving any problem.
7. Working with Others: Students will learn with and from each other.
8. Communicating: Students will discuss their findings by presenting.

This lesson will meet the following Statements of Learning:

1. The student recognises the potential uses of mathematical knowledge, skills and understanding in all areas of learning (statement 15).
2. The student devises and evaluates strategies for investigating and solving problems using mathematical knowledge, reasoning and skills (statement 17).
3. The student describes and explains relationships (statement 16).

9. Flow of the Research Lesson

Steps, Learning Activities Teacher's Questions and Expected Student Reactions	**Teacher Support**	**Assessment**
Introduction		

<table>
<tr>
<td>In the previous lessons you recalled and applied what you learned at Junior Cert about transversals and the angles they make with parallel lines, and about congruent shapes.

What is a transversal?</td>
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<td>Students show that they understand what a transversal means.</td>
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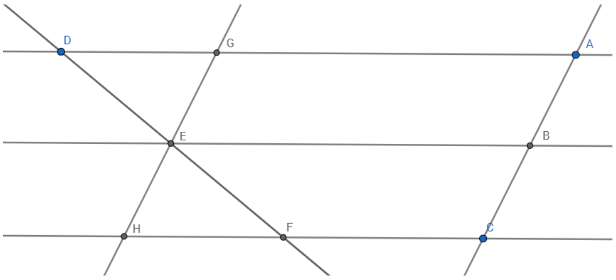
<tr>
<td>Posing Task 1 (5 minutes including introduction)



Line DA is parallel to EB and HC. Line GH is parallel to AB and |AB|=|BC|. Write down other relationships that you can find in the diagram above between:

(a) pairs of angles
(b) pairs of line segments
(c) pairs of triangles.

Give a reason in each case.</td>
<td>The problem will be placed on the board and copies handed out to students.

Students are given 10 minutes to work through the problem.

Since there are so many relationships in part (a) and at most three will be used further on, and since addressing all parts of the task requires a bit of time management, the teacher clarifies the problem by suggesting that students should try to identify 6 relationships in (a), for example.</td>
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<tr>
<td>Student Individual Work (10 minutes)

(a) Pairs of angles
$|\lt \boldsymbol{DEG}| = |\lt \boldsymbol{HEF}|$ vertically opposite
$|\lt GEF| = |\lt DEH|$ vertically opposite
$|\lt \boldsymbol{DGE}| = |\lt \boldsymbol{EHF}|$ alternate
$|\lt \boldsymbol{GDE}| = |\lt \boldsymbol{EFH}|$ alternate
$|\lt DGE| = |\lt EHF|$ alternate
$|\lt EFH| = |\lt BEF|$ alternate
$|\lt GDE| = |\lt BEF|$ corresponding
$|\lt DGE| = |\lt EHF|$ alternate
$|\lt EHF| = |\lt EBC|$ opposite angles in a parallelogram
⋮

(b) Pairs of line segments
$|\boldsymbol{AB}| = |\boldsymbol{GE}|$ opposite sides of a parallelogram
$|\boldsymbol{BC}| = |\boldsymbol{EH}|$ opposite sides of a parallelogram
$|AC| = |GH|$ opposite sides of a parallelogram
$|GA| = |EB|$ opposite sides of a parallelogram
$|EB| = |HC|$ opposite sides of a parallelogram

Deriving a new relationship $|\boldsymbol{GE}| = |\boldsymbol{EH}|$ using the above and $|\boldsymbol{AB}| = |\boldsymbol{BC}|$ which has been given:

$|\boldsymbol{GE}| = |\boldsymbol{EH}|$ because $|AB| = |GE|$, $|BC| = |EH|$ and $|AB| = |BC|$ is given.

(c) Pairs of triangles

ΔDGE and ΔEHF are similar because the corresponding angles are equal.</td>
<td>Teacher circulating the room might ask the students who struggle which angles/line segments appear to be equal and why.

What have you learned about special angles between parallel lines and transversals so far that would help you solve this problem. What do you know about alternate / corresponding angles and can you give an example.

Can you see a shape with equal sides?</td>
<td></td>
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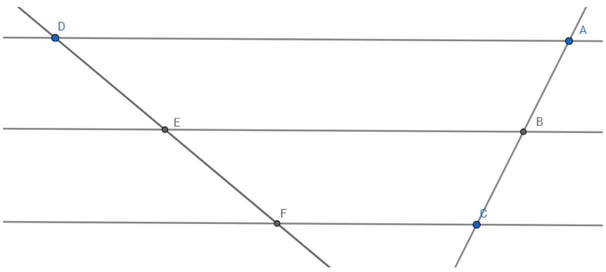
</table>

ΔDGE and ΔEHF are congruent (ASA) because the corresponding angles are equal and the included sides $\|GE\| = \|EH\|$. **New relationships arise due to congruency of triangles**: $\|DE\| = \|EF\|$ and $\|DG\| = \|HF\|$.		
Ceardaíocht /Comparing and Discussing (20 minutes) Ask students to show the class the relationships that they identified and discuss the reasons for each. Since there is a significant number of equal angles, invite students to show 6-8 pairs on the board ensuring that the relationships crucial for congruency of triangles are listed. Help students realise that through deduction new relationships can be derived from the existing ones.	Listing more relationships than required in the actual proof of Theorem 11 will help students accept as a natural consequence of investigation that there exist relationships that might be redundant and one needs to understand which relationships are useful and select the most suitable ones to complete Task 2 (a formal proof of Theorem 11).	Students articulate why angles/line segments are equal providing a clear reason.
Summing up Task 1 (3 minutes) Looking back at the opening problem, what did we start with? Three parallel lines and three transversals one of which had equal segments cut off by the parallel lines. What new relationships have you obtained through your investigation of angles, line segments and triangles? We used parallelograms to derive |GE|=|EH| and congruent triangles to derive |DE|=|EF|. What can you conclude about the transversals cut by three parallel lines? That they cut off equal segments on transversals. Help students articulate the main result of this activity: *If three parallel lines cut off equal segments on one transversal then they cut off equal segments on other transversals.*	Once students articulate the statement of Theorem 11, a poster with the statement is placed on the board.	
Posing Task 2 (2 minutes)  Given are three parallel lines and two transversals DF and AC with |AB|=|BC|. What is the relationship between |DE| and |EF|? How can you prove it?	Alternatively, a diagram with three parallel lines and one transversal AC could be given with |AB|=|BC|. Then students could be asked to draw the DF transversal wherever they want and prove that |DE|=|EF|. Students are given 10 minutes to work through Task 2.	All students should be able to pose a hypothesis that |DE|=|EF|. **Important**: If time does not allow to solve Task 2 (proof of Theorem 11) within the same lesson, the teacher gives it as homework. Solutions are discussed in the follow up lesson during which students collate and structure the material from the previous lesson to derive a proof of Theorem 11.

<table>
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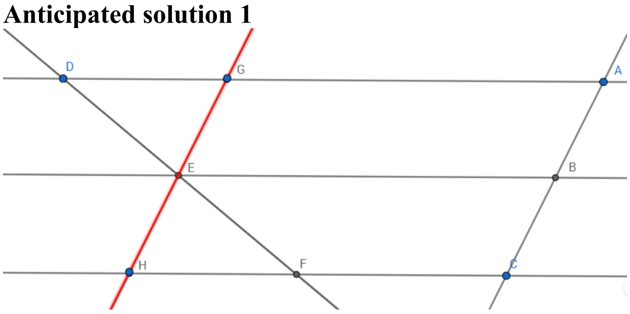
<td>Students individual work (10 minutes)

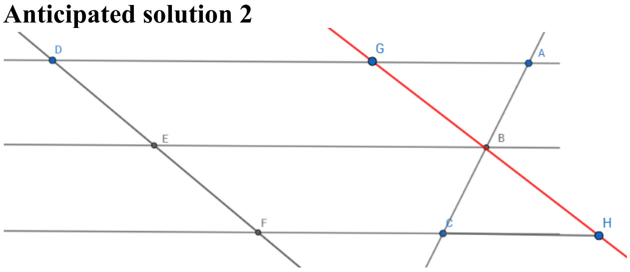
Anticipated solution 1


Draw the line GH parallel to AC.
$|\angle DEG| = |\angle HEF|$ vertically opposite
$|\angle DGE| = |\angle EHF|$ alternate

$|AB| = |GE|$ opposite sides of a parallelogram
$|BC| = |EH|$ opposite sides of a parallelogram

Since $|AB| = |BC|$, it follows that $|GE| = |EH|$.

By ASA, ΔDGE and ΔEHF are congruent. Hence $|DE| = |EF|$.

Anticipated solution 2


Draw the line GH parallel to DF.
$|\angle ABG| = |\angle CBH|$ vertically opposite
$|\angle GAB| = |\angle BCH|$ alternate
Since $|AB| = |BC|$, it follows from ASA, ΔGAB and ΔCBH are congruent. Hence $|GB| = |BH|$.
Since
$|DE| = |GB|$ opposite sides of a parallelogram
$|EF| = |BH|$ opposite sides of a parallelogram, we conclude that $|DE| = |EF|$.</td>
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<td></td>
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<tr>
<td>Discussion (15-20 minutes)
Let's see how you can use what you have learned from Task 1 to prove your statement in Task 2.

What did you do that helped you prove that |DE|=|EF|?

Help students realise the need to draw another transversal and that not all relationships derived in Task 1 are needed to prove the statement.

By discussing the two approaches students should realise that there are multiple approaches to a formal proof and that they do not need to memorise the steps but instead use the relationships that they identify.</td>
<td>Are students able to select the most relevant pieces from Task 1 in order to solve Task 2?</td>
<td>Can students see the need to draw another transversal (parallel to AC or DF, other transversals can be considered as well, for example a perpendicular to the parallel lines).

Do students notice that not all the relationships spotted in Task 1 are needed?</td>
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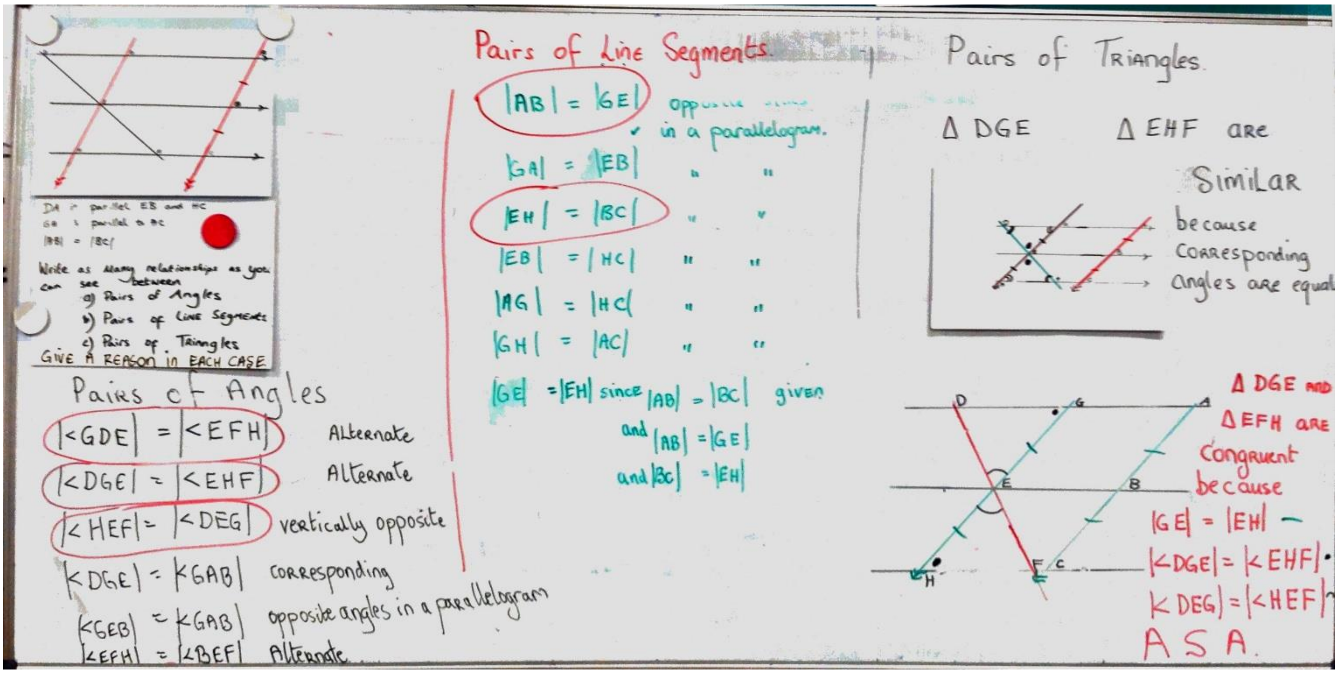
Summing up & Reflection (5 minutes) Using the writing on the board, review what students learned through the lesson. Ask students to write a short reflection about what they learned. Assessment activity 1: Are the three parallel lines in Tasks 1 and 2 equally spaced? Assessment activity 2: Using what you have learned about three parallel lines and transversals show that, if five parallel lines cut off equal segments on the same transversal, they will cut off equal segments on any other transversal.	Assessment activities can be used as homework or in a follow up lesson.	Each student summerises their learning.

10. Board Plan



11. Evaluation

The effectiveness of the lesson will be discussed using the following questions:

a) *Were the goals met? (Were students able to identify the relationships and arrive at the statement of the theorem? Did students gain more confidence in dealing with a formal proof? Did students show positive attitude towards the concept of a theorem and its proof?)*

b) *What methods did students use?*

c) *What questions/comments did students have?*

d) *What were the common misconceptions and misunderstandings?*

e) *How and when did students' understanding change?*

f) *Did students' presentation and discussion promote their thinking and learning?*

g) *Was the flow of the lesson coherent?*

h) *Did the students display a positive disposition?*

i) *Did the activities support the goals?*

Worksheets and photographs of students' work will be collected.

12. Reflection

The lesson was an excellent opportunity for the students to reflect and to apply prior knowledge. Our objectives were that students would engage with the lesson material and have a positive attitude towards geometry and develop their confidence in their deduction skills. All students were able to identify the relationships between pairs of angles and pairs of line segments, but not many noticed similar or congruent triangles. Formal statement of the theorem was articulated by one of the students by the end of the lesson. We feel that the lesson really fostered a positive mind-set regarding the concept of geometrical proofs. Even the shyer, more relaxed students had their sheet fully filled by end of lesson and engaged with the process; they made notes of concepts raised and addressed those that they did not themselves discover.

We found that the students engaged fully with the lesson material and were keen to take on the challenge. They showed a very positive attitude to the task in hand and were eager to participate and discuss their results. Many students took the initiative to make their own constructions on the diagram to help them draw their own conclusions. The use of colour was also evident in their work as they were examining the various relationships. It was observed that some students had difficulties in labelling and notations of angles and lines. One student made a mistake at the board in explaining his answer regarding the correct notation to identify angles. This in turn opened up the questioning aspect of the lesson and students became more questioning in nature.

Student understanding changed when they moved onto the pairs of equal line segments. One student asked the student presenter "*How do you know it's a parallelogram?*". This led to students thinking about the concept of a parallelogram and what constitutes a parallelogram.

There were some misconceptions regarding the grounds for congruency in the task at hand. Some students thought that SAS was better than SSS and AAA for proving congruency, which provides evidence of their misconception. This misconception was addressed by a student who challenged another student's contribution that AAA is sufficient to promote congruency. Lesson showed that students know what congruency means but not the conditions which give rise to congruency. The above observations show that students' presentation and discussion promoted their thinking and learning. Due to the shortcomings regarding congruent triangles and properties of parallelograms there was no time to discuss Task 2 which was given as homework.

Nevertheless, we think that the lesson flowed very well, with a clear beginning, middle and end. The division of the worksheet clearly delineated the divisions between the three sections of the lesson. It ensured that the students did not spend all their time discussing the angles. Board work greatly facilitated the flow of the lesson; very clear and still accessible at the end of the lesson. The longer time period for the lesson was very valuable. No students were totally confused or lost; any misconceptions were addressed by their own discussion. **This lesson raises interesting comparisons with how we normally teach the theorems.**

Recommendations: When one student articulated the formal statement of the theorem we realised that more attention should be drawn to this on the board. Perhaps coloured pre-prepared posters/stickers could be used as an extra teaching tool to highlight crucial points in the lesson.

In future, more time needs to be spent prior to the lesson to ensure students fully understand the conditions of congruency and properties of parallelograms.