Further Exploration of Patterns

Project Maths
Tionscadal Mata
Development Team

Abstract Quadratic Patterns

	$a+b+c$	$4a+2b+c$	$9a+3b+c$	$16a+4b+c$	$25a+5b+c$
1st change	$3a+b$	$5a+b$	$7a+b$	$9a+b$	
2nd change	$2a$	$2a$	$2a$		

Coefficient of n^2 is $\frac{1}{2}(2a) = a$.

Constant second change, therefore it is quadratic starting with an^2.

Write out the an^2 series a, $4a$, $9a$, $16a$,... and subtract from the original series.

$$\begin{array}{lccccc} & a+b+c, & 4a+2b+c, & 9a+3b+c, & 16a+4b+c, & 25a+5b+c,\ldots \\ - & a & 4a & 9a & 16a & 25a,\ldots \\ \hline & b+c & 2b+c & 3b+c & 4b+c & 5b+c,\ldots \end{array}$$

1st change: b b b b — This is linear bn

If you now go down form the 1st term which is $(b+c)-b=c$

The General Term is now an^2+bn+c

Generalising Quadratic Sequences

n^2	1,	4,	9,	16,	25,…
1^{st} change	3	5	7	9	
2^{nd} change	2	2	2		

$2n^2$	2,	8,	18,	32,	50,…
1^{st} change	6	10	14	18	
2^{nd} change	4	4	4		

When you get half the 2^{nd} change you get the coefficient of n^2

$4n^2$	4,	16,	36,	64,	100,…
1^{st} change	12	20	28	36	
2^{nd} change	8	8	8		

$0.5n^2$	0.5,	2,	4.5,	8,	12.5,…
1^{st} change	1.5	2.5	3.5	4.5	
2^{nd} change	1	1	1		

Your Turn [2.7 pg. 18]

Write the General Formula for the following paterns.

(a) 3, 12, 27, 48, 75,…

(b) 0.25, 1, 2.25, 4, 6.25,…

Solutions:

(a) $3n^2$ **(b)** $0.25n^2$

More Difficult Quadratic Patterns – Method 1

6, 12, 20, 30, 42,…

1st change 3 5 7 9

2nd change 2 2 2 Therefore it is n^2 with other terms.

Write out the n^2 series 1, 4, 9, 16,… and subtract from the original series.

	6,	12,	20,	30,	42,…
–	1,	4,	9,	16,	25,…
	5,	8,	11,	14,	17,…

1st change 3 3 3 3 This is linear $3n$

If you now go down form the 1st term which is $5-3=2$

The general term is now n^2+3n+2

More Difficult Quadratic Patterns – Method 2

$$6,\ 12,\ 20,\ 30,\ 42,\ldots$$

Look for lowest common factor, here it is 2

$$2\times 3,\ 3\times 4,\ 4\times 5,\ 5\times 6,\ 6\times 7,\ldots$$

This is an $AP \times AP$

First AP $\quad 2,\ 3,\ 4,\ 5,\ 6,\ldots \quad T_n = a+(n-1)d = 2+(n-1)1 = n+1$

Second AP $\quad 3,\ 4,\ 5,\ 6,\ 7,\ldots \quad T_n = a+(n-1)d = 3+(n-1)1 = n+2$

$$T_n \times T_n = (n+1)(n+2) = n^2+3n+2$$

Your Turn [2.8 pg. 18]

Write the General Formula for the following paterns.

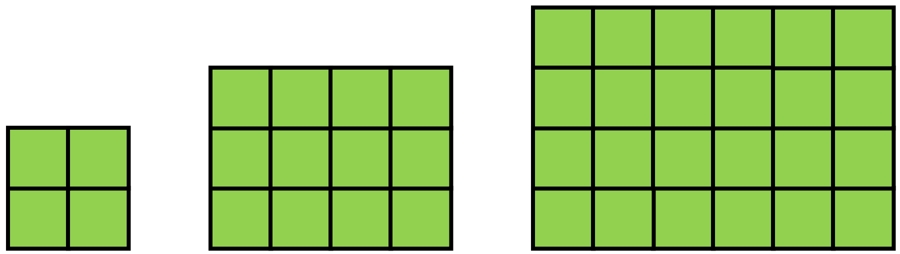
(a) 5, 12, 21, 32, 45, ...

(b) 5, 15, 31, 53, 81, ...

Solutions:

(a) $n^2 + 4n$ **(b)** $3n^2 + n + 1$

Your Turn [2.9 pg. 18]



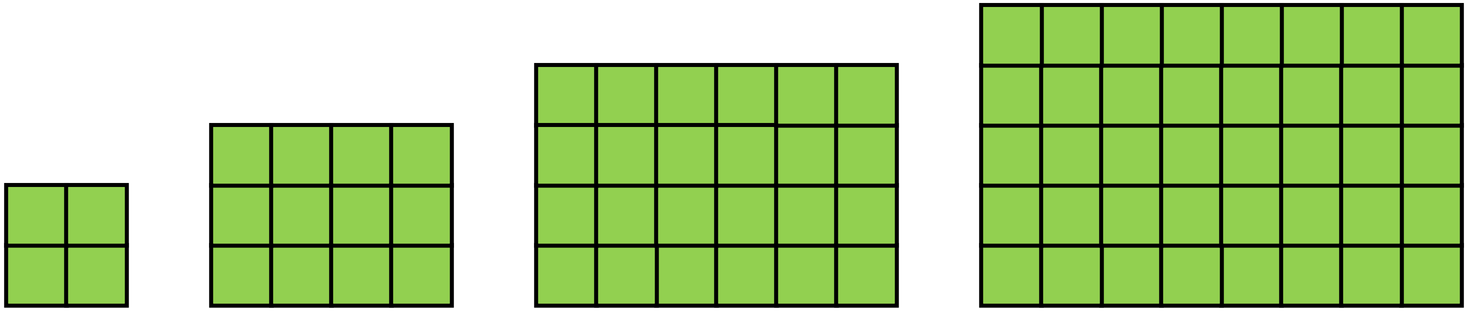
1^{st} 2^{nd} 3^{rd}

How many blocks are in the 4^{th} pattern?

Write a gerneral formula to find the number of in the n^{th} pattern.

How many blocks are in the 8^{th} pattern?

SOLUTION



1st 2nd 3rd 4th

	4	12	24	40
1^{st} change	8	12	16	
2^{nd} change	4	4	Constant, therefore quadratic	

Method 1

	4	12	24	40
1st change	8	12	16	
2nd change	4	4	2	

Therefore it is $2n^2$ with other terms.

Write out the $2n^2$ series 2, 8, 18, 32,... and subtract from the original series.

$$\begin{array}{rrrrr} & 4, & 12, & 24, & 40,\ldots \\ - & 2, & 8, & 18, & 32,\ldots \\ \hline & 2, & 4, & 6, & 8, \end{array}$$

1st change 2 2 2 This is linear $3n$

If you now go down form the 1^{st} term which is $2-2=0$

The general term is now $2n^2+2n$

$T_8=2(8)^2+2(8)=144$ blocks

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Method 2

4, 12, 24, 40,

Look for lowest common factor, here it is 2 [Looking at length $\times$ breadth]

$2\times2,\ 3\times4,\ 4\times6,\ 5\times8,$

This is an AP $\times$ AP

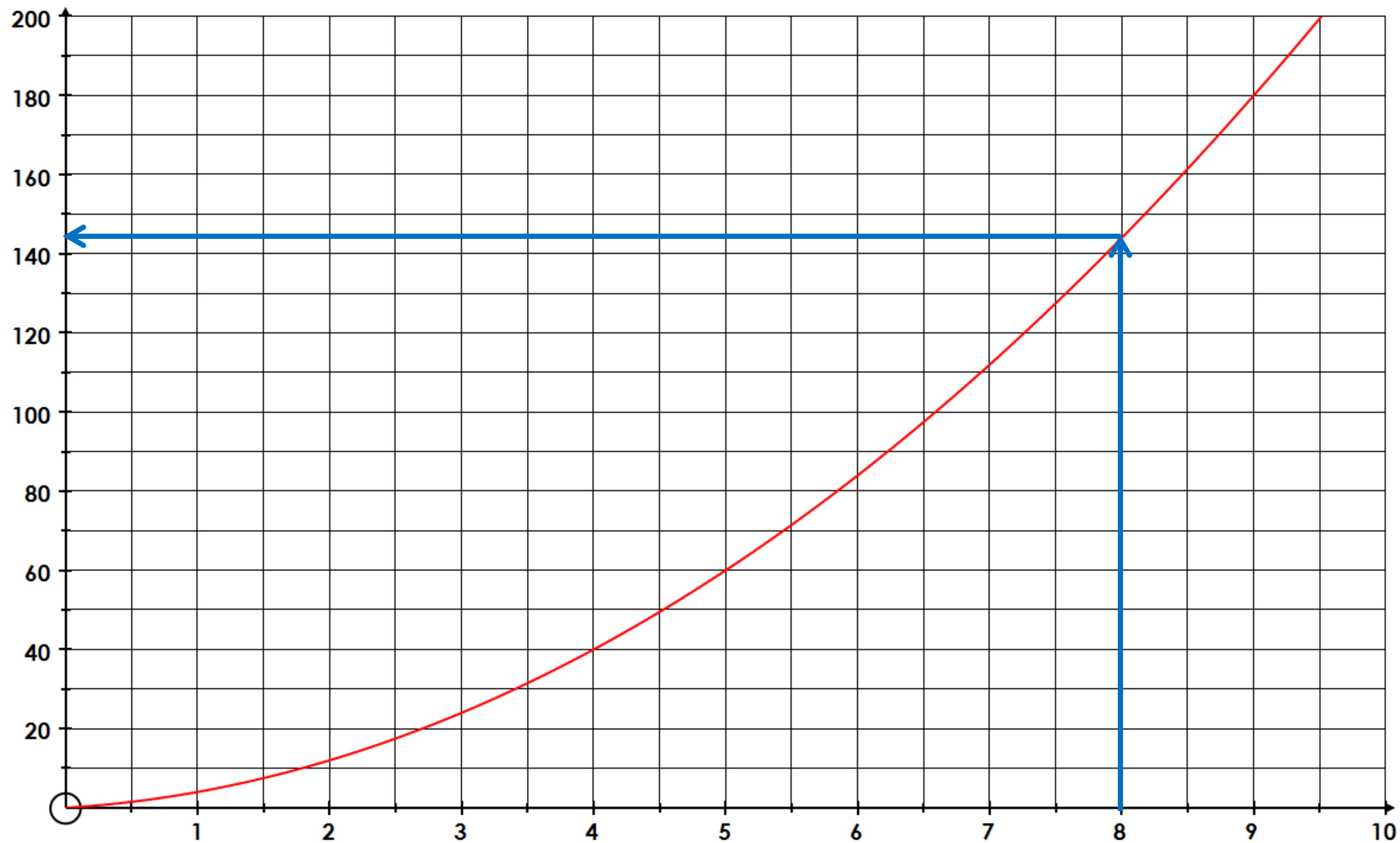
First AP 2, 3, 4, 5,... $T_n = a+(n-1)d = 2+(n-1)1 = n+1$

Second AP 3, 4, 5, 6, 7,... $T_n = a+(n-1)d = 2+(n-1)2 = 2n$

$T_n \times T_n = (n+1)(2n) = 2n^2 + 2n$

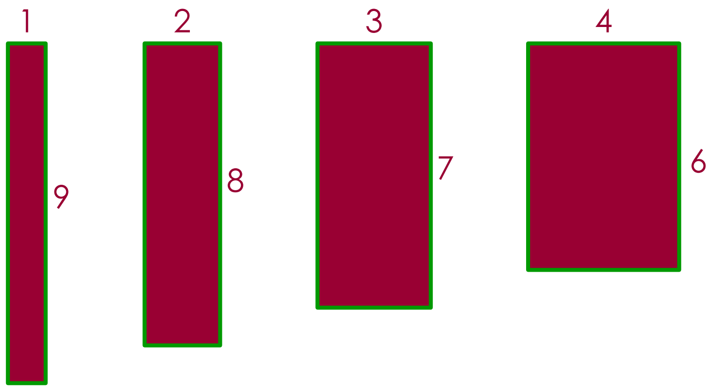
$T_8 = 2(8)^2 + 2(8) = 144$ blocks

Graphing the Couples



$T_8 = 144$

Given you have 20 metres of wire, what is the maximum rectangular shaped area that you can enclose?



Area	9	16	21	24	
1^{st} change	7	5	3		
2^{nd} change		-2	-2		Therefore it is $-n^2$ with other terms.

Write out the $-n^2$ series $-1,\ -4,\ -9,\ -16, \ldots$ and subtract from the original series.

$$\begin{array}{rrrrr} & 9, & 16, & 21, & 24,\ldots \\ - & -1, & -4, & -9, & -16,\ldots \\ \hline & 10, & 20, & 30, & 40, \end{array}$$

This is linear $10n$

The general term for the area is $-n^2 + 10n$

Geometric

$T_1 = 2$

$T_2 = 4$

$T_3 = 8$

$T_4 = 16$

$\vdots$

$T_n = ?$

This is a Geometric Sequence with $T_1 = 2 = a$

Each term is multiplied by 2 to get the next term: $r(\text{Common Ratio}) = 2$

$$\frac{T_n}{T_{n-1}} = r$$

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Finding the T_n of a Geometric Sequence

$$T_1 = a$$

$$T_2 = ar$$

$$T_3 = (ar)r = ar^2$$

$$T_4 = (ar^2)r = ar^3$$

$$T_5 = (ar^3)r = ar^4$$

$$\vdots$$

$$T_n = ar^{n-1}$$ Exponential

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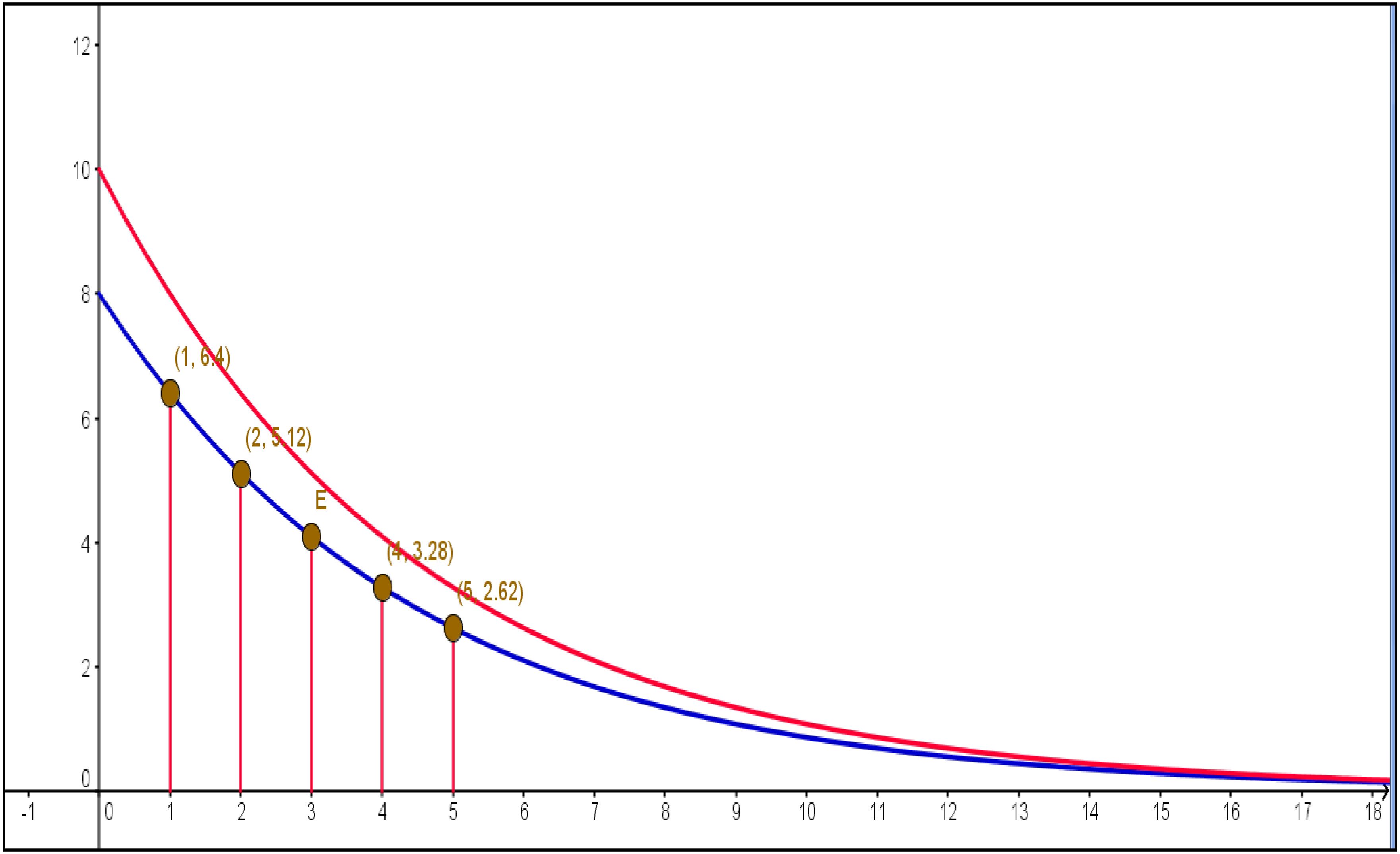
A ball is dropped from a height of 8 m. The ball bounces to 80% of its previous height with each bounce. How high (*to the nearest cm*) does the ball bounce on the fifth bounce.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
8,	6.4,	5.12,	4.96,	3.2768,	2.62144

2.62144m = 262 cm

Is this a Linear, Quadratic or Cubic pattern?

Let's look at the graph of this.



Blue: $f(x) = 8(0.8)^x$ or $T_n = 8(0.8)^n$

Write down the function which describes the red graph.
What is the total distance travelled by the ball when it hits the ground for the 5th time?

What if we were asked to find the total distance travelled when the ball hits the ground for the 20th time. Is there any general way of doing it?

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad S_n = \frac{a(1 - r^n)}{1 - r}$$

Finding the S_n of a Geometric Series

$$S_n \quad = a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

$$rS_n \quad = \quad ar + ar^2 + ar^3 + ar^4 + \cdots \quad + ar^n$$

$$S_n - rS_n = a + 0 + 0 + 0 + 0 + \cdots$$

Subtracting

$$(1-r)S_n = a - ar^n$$

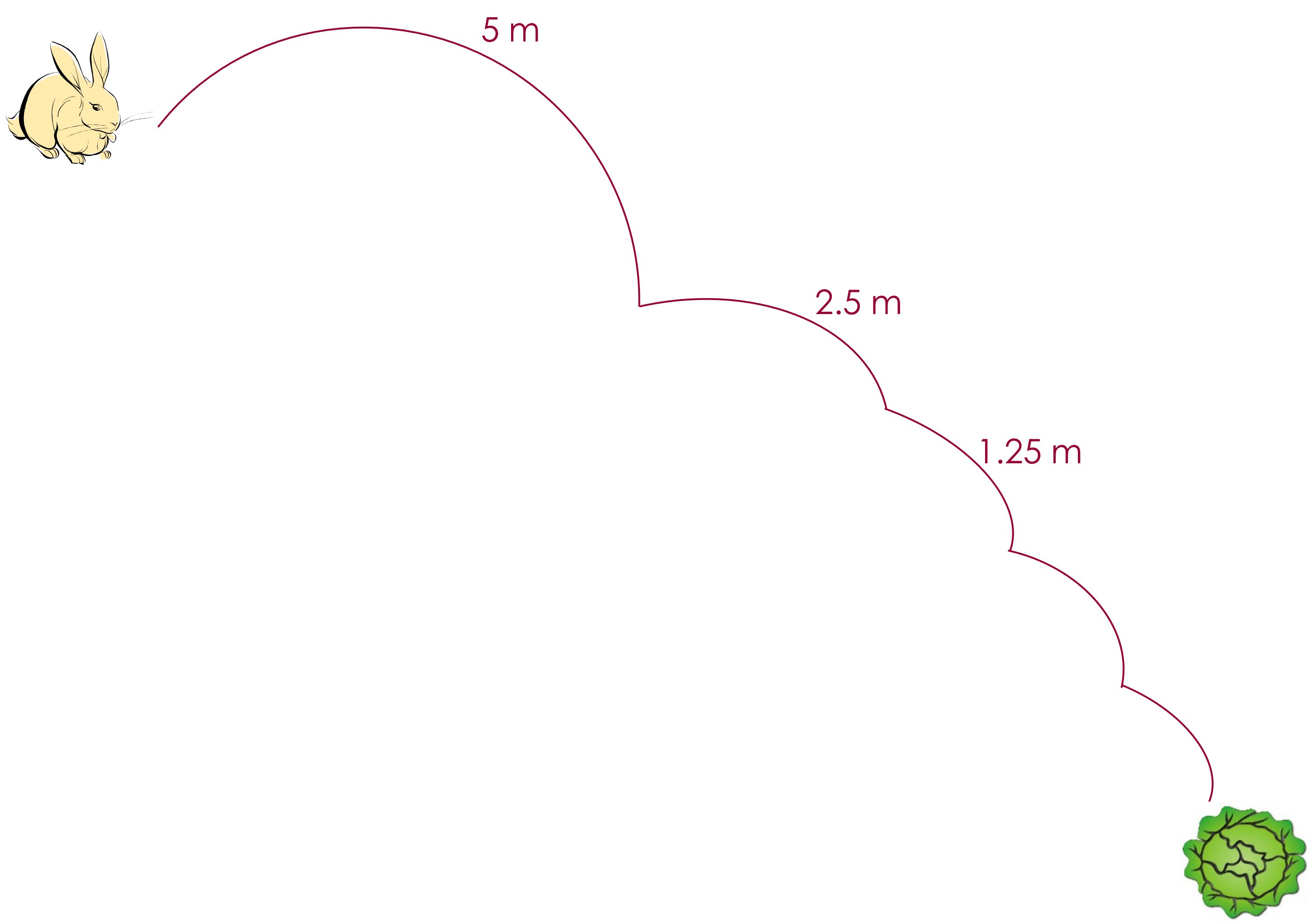
$$S_n = \frac{a - ar^n}{1-r}$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r-1}$$

These formulas can also be proved by Induction

Link to Student's CD

A rabbit is 10 metres away from a some food. It hops 5 metres, then hops 2.5 metres, then 1.25 metres, and so on, hopping half its previous hop each time. What will the length of the 6^{th} hop be?

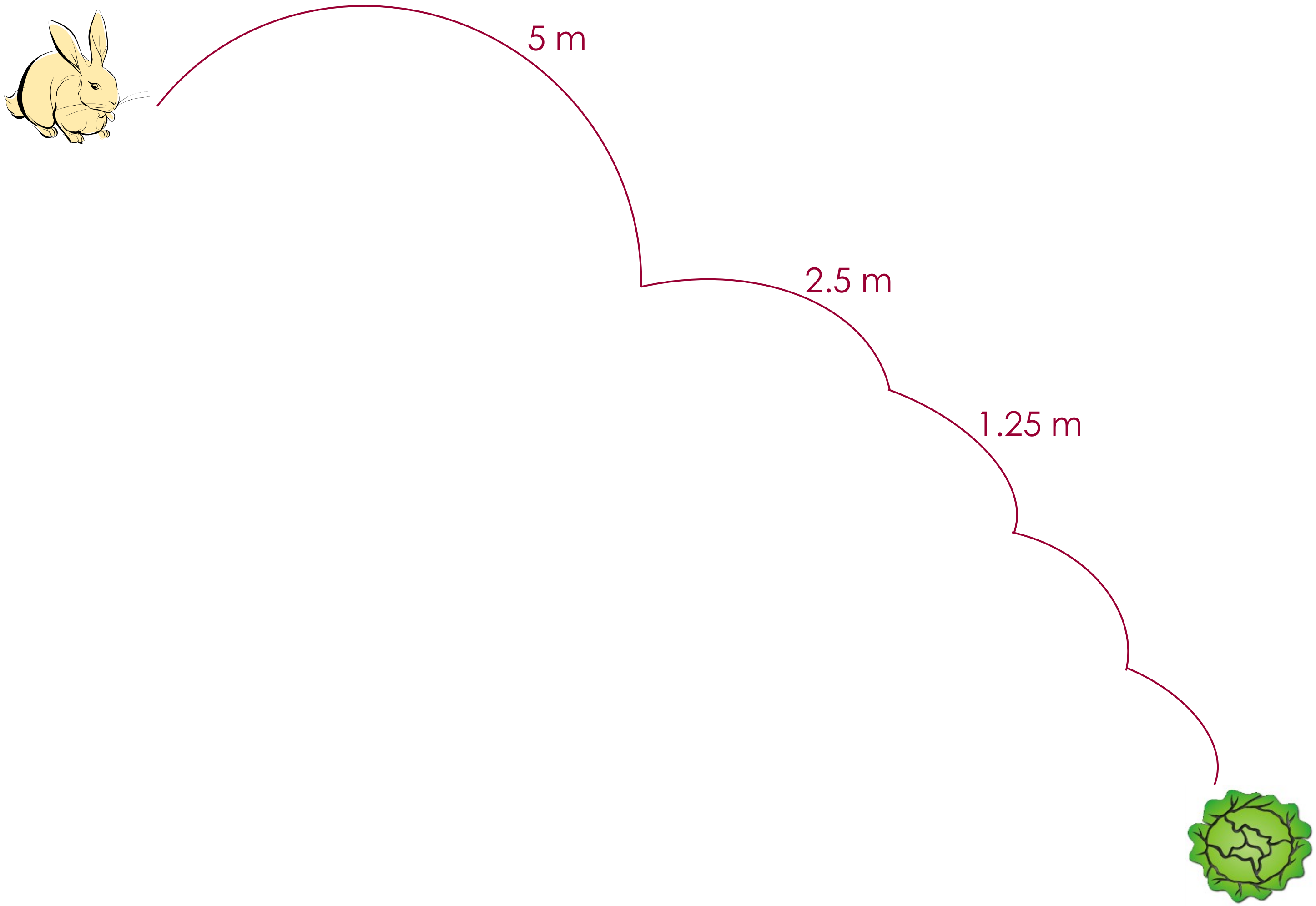


5, 2.5, 1.25, 0.625, 0.3125,…

What type of pattern is this? Discuss

A rabbit is 10 metres away from some food. It hops 5 metres, then hops 2.5 metres, then 1.25 metres, and so on, hopping half its previous hop each time.

If the rabbit kept hopping forever, what in theory would be the total distance travelled by it?



The Sum to Infinity of a GP

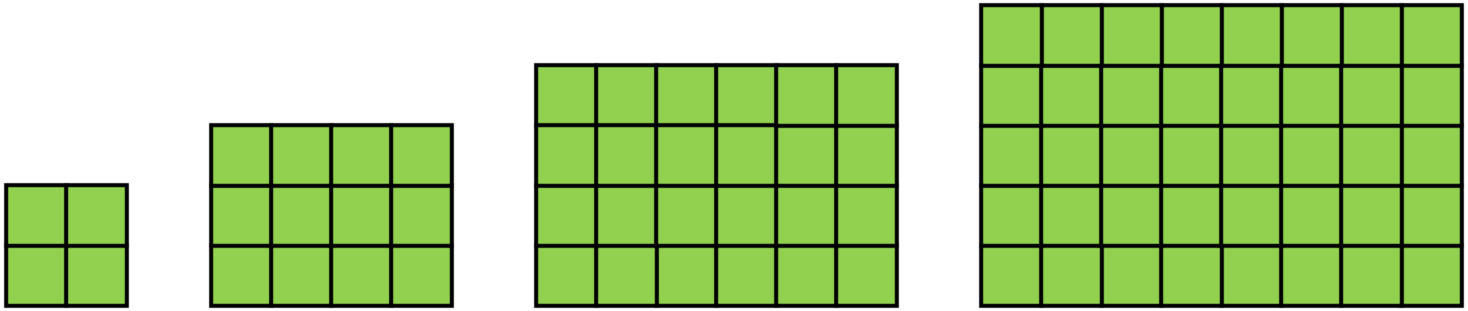
For $|r| < 1 \qquad r^{\infty} = 0$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$S_{\infty} = \frac{a}{1-r} - \frac{0}{1-r} \qquad as\ r^{\infty} = 0 \ \text{ for } |r| < 1$$

$$S_{\infty} = \frac{a}{1-r} \ \text{ for } \ |r| < 1$$

Extending the Blocks Question



Find the total number of blocks required to make the first 25 patterns.

$$T_n = 2n^2 + 2n$$

$$S_n = 2\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$S_n = 2\left[\frac{n(n+1)(2n+1)}{6}\right] + 2\left[\frac{n(n+1)}{2}\right]$$

$$S_n = \left[\frac{n(n+1)(2n+1)}{3}\right] + n(n+1)$$

$$S_{25} = \left[\frac{25(25+1)(2(25)+1)}{3}\right] + 25(25+1)$$

$$S_{25} = 11{,}700$$

Three Formulae

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \qquad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$$

These formulas can be proved by Induction

Summary of GP formulae

$$T_n \text{ of a GP} = ar^{n-1}$$

$$S_n \text{ of an GP} = \frac{a(r^n - 1)}{r - 1} \quad \text{for } r > 1 \quad \text{or} \quad \frac{a(1 - r^n)}{1 - r} \quad \text{for } r < 1$$

$$T_n = S_n - S_{n-1}$$

$$S_\infty = \frac{a}{1 - r} \quad \text{for } |r| < 1$$

Express 1.: orm of $\frac{a}{b}$ where a and $b \in \mathbb{N}$

Method 1	**Method 2**
1.: 222222…	Let $x = 1.222222222\ldots$
$= 1 + 0\cdot 2 + 0\cdot 02 + 0\cdot 002 + 0\cdot 0002 + 0\cdot 00002 + \cdots$	$10x = 12.22222222\ldots$
$= 1 + \left[\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \cdots\right]$	$x = 1.222222222\ldots$
This is an infinite GP with $a = \frac{2}{10}$ and $r = \frac{1}{10}$	$9x = 11$
$S_\infty = \frac{a}{1-r}$	$\therefore x = \frac{11}{9}$
$S_\infty = \frac{\frac{2}{10}}{1-\frac{1}{10}} = \frac{2}{9}$	
$\therefore 1.$: 9 $= \frac{11}{9}$	

Express 1. … form of $\frac{a}{b}$ where a and $b \in \mathbb{N}$

Method 1	**Method 2**

Method 1

$1.\ \ldots 434343\ldots$

$= 1 + 0\cdot 43 + 0\cdot 0043 + 0\cdot 000043 + \cdots$

$= 1 + \left[\frac{43}{100} + \frac{43}{10000} + \frac{43}{1000000} + \cdots\right]$

This is an infinite GP with $a = \frac{43}{100}$ and $r = \frac{1}{100}$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\frac{43}{100}}{1-\frac{1}{100}} = \frac{43}{99}$$

$\therefore 1.\ \ldots\ 99 = \frac{142}{9}$

Method 2

Let $x = 1.43434343\ldots$

$100x = 143.43434343\ldots$

$x = 1.4343434343\ldots$

$99x = 142$

$\therefore x = \frac{142}{99}$

Other Types of Series

$$\frac{1}{AP \times AP}$$

Show that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Fill in the various values in the square brackets and find the sum to n terms of the series whose T_n is $\dfrac{1}{n(n+1)}$

Find the sum of the first 20 terms of the series

$$T_1 = \frac{1}{[\ \ ]} - \frac{1}{[\ \ ]}$$

$$T_2 = \frac{1}{[\ \ ]} - \frac{1}{[\ \ ]}$$

$$T_3 = \frac{1}{[\ \ ]} - \frac{1}{[\ \ ]}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T_{n-1} = \frac{1}{[\ \ ]} - \frac{1}{[\ \ ]}$$

$$T_n = \frac{1}{[\ \ ]} - \frac{1}{[\ \ ]}$$

$S_n =$

Solution

$$T_1 = \frac{1}{1} - \frac{1}{2}$$

$$T_2 = \frac{1}{2} - \frac{1}{3}$$

$$T_3 = \frac{1}{3} - \frac{1}{4}$$

$$\vdots \quad \vdots \quad \vdots$$

$$T_{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$T_n = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$S_{20} = 1 - \frac{1}{20+1}$$

$$S_{20} = \frac{20}{21}$$

Arithmethico Geometric AP x GP

$$1+2r+3r^2+\cdots$$

Find the T_n of the following sequence 2, 8, 24, 64, 160…

Find the T_n of the following sequence $1\times 2,\ 2\times 4,\ 3\times 8,\ 4\times 16,\ 5\times 32\ldots$

Each term in this sequence is an $AP \times GP$

T_n of $AP = n$

T_n of $GP = 2(2)^{n-1} = 2^n$

Combined, $T_n = n(2^n)$

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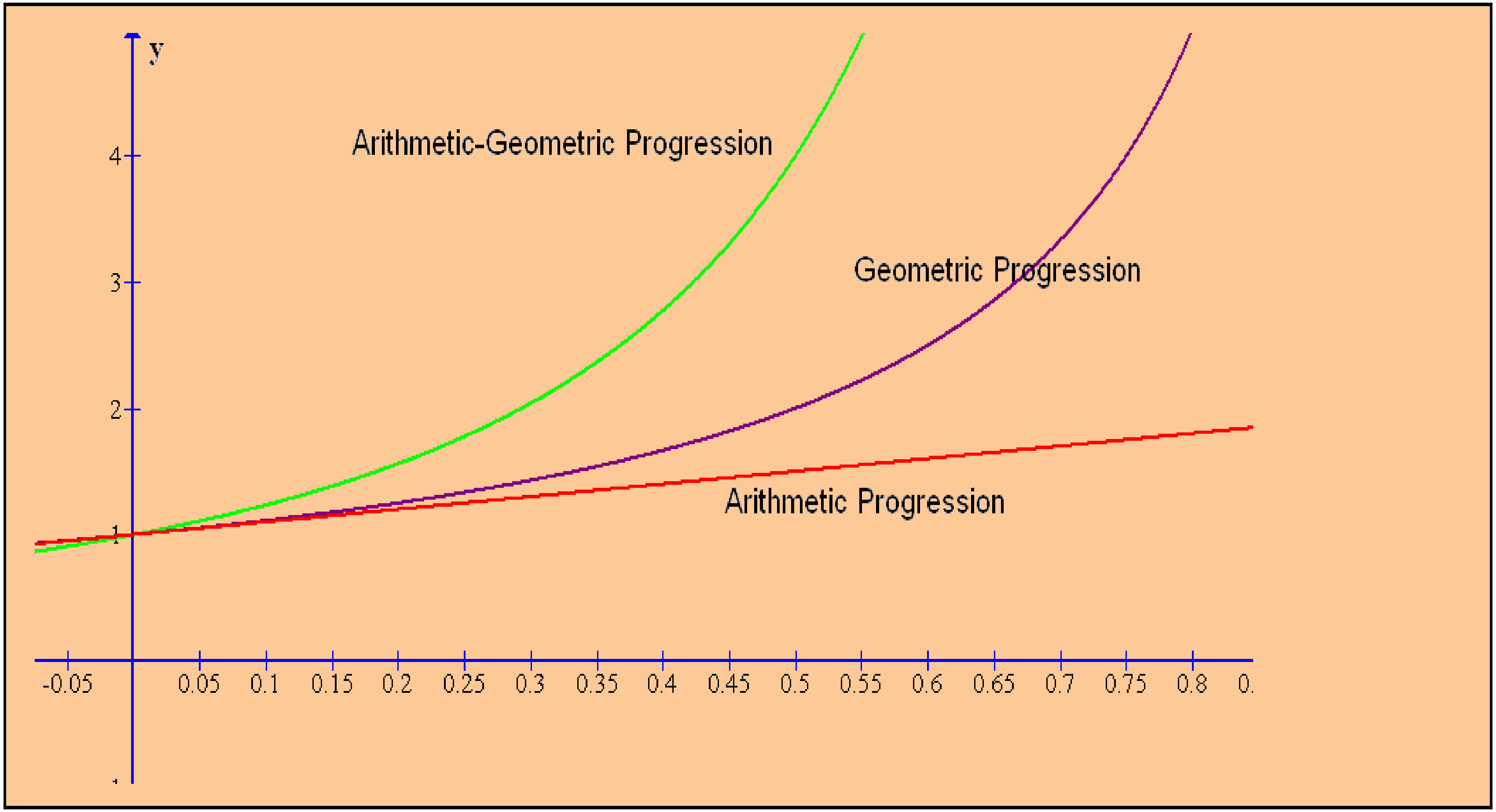

GeoGebra Three Graphs
Function Inspector

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2012 LCHL Q4

In a science experiment, a quantity $Q(t)$ was observed at various points in time t. Time is measured in seconds from the instant of the first observation. The table below gives the results.

t	0	1	2	3	4
$Q(t)$	2·920	2·642	2·391	2·163	1·957

Q follows the rule of the form $Q(t) = Ae^{-bt}$, where A and b are constants.

(a) Use any two of the observations from the table to find the value of A and the value of b, correct to three decimal places.

(b) Use a different observation from the table to verify your values for A and b.