Nix the ~~Tricks~~

A guide to avoiding shortcuts that cut out math concept development

by

Tina Cardone

and

the online math community known as the MTBoS

NIX THE TRICKS

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Tina Cardone and the MTBoS

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All student work was collected by classroom teachers during the course of regular lessons, then submitted to MathMistakes.org. To protect the privacy of students (and in many cases to improve legibility), each example was re-written in the author's handwriting.

"I would say, then, that it is not reasonable to even mention this technique. If it is so limited in its usefulness, why grant it the privilege of a name and some memory space? Cluttering heads with specialized techniques that mask the important general principle at hand does the students no good, in fact it may harm them. Remember the Hippocratic oath - First, do no harm."

-Jim Doherty

Contents

Preface

In the beginning, there was a calculus teacher complaining about students' lack of a definition of slope. Then there was a conversation among my department members on tricks we hate seeing kids show up to our classes with. I expanded the conversation to members of my online math community. We brainstormed and debated what constituted a trick and which were the worst offenders. I organized. More people joined in on the conversation and shared better methods to emphasize understanding over memorization. I organized some more. Contributions started to slow down. The end result was 17 pages in a google doc. I had grand dreams of a beautifully formatted resource that we could share with teachers everywhere. A few people shared my dream. We discussed formatting and organization and themes. Finally, inspired by NaNoWriMo, I opened up LaTeX and started typesetting. I got some help. This document was born. I hope you enjoy it all the more knowing how many people's ideas are encapsulated here.

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Each chapter follows a concept thread. For example, you might see how content knowledge should build from drawing representations of fractions to solving proportions. Feel free to read this as a book, from front to back, or jump directly to those sections that apply to your grade level of interest.

Chapter 1

Introduction

This text is inspired by committed teachers who want to take the magic out of mathematics and focus on the beauty of sense-making. It is written for reflective teachers who embrace the Standards for Mathematical Practice. The contributors are people who wish for teachers everywhere to seek coherence and connection rather than offer students memorized procedures and shortcutting tricks. Students are capable of rich conceptual understanding; do not rob them of the opportunity to experience the discovery of new concepts.

This is a hard step to take, students will have to think and they will say they do not want to. Students (and parents and tutors) will need to readjust their expectations; but it is in the best interest of students everywhere to make the focus of mathematics critical thinking. Will you help math reclaim its position as a creative and thought-provoking subject?

"But it's just to help them remember - I taught them the concept!"

SOH CAH TOA is a mnemonic device. There is no underlying reason why sine is the ratio of opposite to hypotenuse, it is a definition. Kids can use this abbreviation without losing any understanding.

FOIL is a trick. There is a good reason why we multiply binomials in a certain way, and this acronym circumvents understanding the power of the distributive property. If you teach the distributive property, have students develop their own shortcut and then give it a name, that is awesome. However, the phrase "each by each" is more powerful since it does not imply that a certain order is necessary (my honors PreCalc students were shocked to hear OLIF would work just as well as FOIL) and reminds students what they are doing. Many students will wait for the shortcut and promptly forget the reason behind it if the trick comes too soon.

"My students can't understand the higher level math, but they do great with the trick."

If students do not understand, they are not doing math. Do not push students too far, too fast (adolescent brains need time to develop before they can truly comprehend abstraction), but do not sell your students short either. The world does not need more robots; asking children to mindlessly follow an algorithm is not teaching them anything more than how to follow instructions. There are a million ways to teach reading and following directions, do not reduce mathematics to that single skill. Allow students to experience and play and notice and wonder. They will surprise you! Being a mathematician is not limited to rote memorization (though learning the perfect squares by heart will certainly help one to recognize that structure). Being a mathematician is about critical thinking, justification and using tools of past experiences to solve new problems. Being a successful adult involves pattern finding, questioning others and perseverance. Focus on these skills and allow students to grow into young adults who can think; everything else will come in time.

"This is all well and good, but we don't have enough TIME!"

Yes, this approach takes an initial investment of time. I would argue that we should be pushing back against the testing pressure rather than pushing back against concept development, but that is a whole other book. First, if we teach concepts rather than tricks, students will retain more information. Each year will start with less confusion of "keep-change-change or keep-keep-change?" and any associated misconceptions, which means each year will have more time for new content. Second, teachers have to make tough choices all the time, and this is another one. Every year my department gets together and tries to guess which topics in Geometry are least likely to be on the state test. We leave those topics for after the exam and do our very best to teach the other topics well. If students go into the test with solid reasoning skills they will at the very least be able to determine which choices are reasonable, if not reason all the way through an inscribed angle problem (circles usually lose our lottery).

Read through these pages with an open mind. Consider how you can empower students to discover a concept and find their own shortcuts (complete with explanations!). I do not ask you to blindly accept these pages as fact any more than I would ask students to blindly trust teachers. Engage with the content and discover the best teaching approaches for your situation. Ask questions, join debates and make suggestions at NixTheTricks.com or share and discuss with your colleagues.

Chapter 2

Operations and Algebraic Thinking

2.1 Nix: Add a Zero (Multiplying by 10)

Because:

$$10 \cdot 10 = 100$$

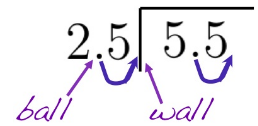
When students are studying integers multiplying by 10 means to "add a zero" but once they head into the realm of real numbers the phrase changes to "move the decimal point." Neither phrase conveys any meaning about multiplication or place value. "Add a zero" should mean "add the additive identity" which does not change the value at all!

Fix:

Have students multiply by ten just like they multiply by any other number. When a student discovers a pattern: first congratulate them for noticing and sharing, then ask them why they think their pattern will always work and finally remind them to be careful about mindlessly applying it in novel situations (ask them about $10 \cdot 0$ for example). One way to help students see why their pattern works is to use a place value chart to show what is happening: each of the ones becomes a ten ($1 \cdot 10 = 10$), each of the tens becomes a hundred ($10 \cdot 10 = 100$), etc. The pattern of shifting works all up and down the decimal scale. You could name the pattern "Student A's rule for multiplying by 10" if you want to be able to refer back to it.

2.2 Nix: Ball to the Wall

Because:



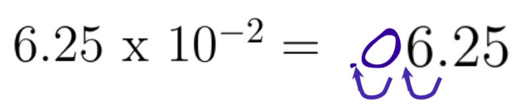
Students have no idea why they are moving the decimal point, which means they are likely to misinterpret this rule and think that $5.5 \div 2.5 = 5.5 \div 25$ rather than the correct statement $5.5 \div 2.5 = 55 \div 25$.

Fix:

The decimal means there is only part of a whole. Since it is easier to work with whole numbers, point out that we can create a different question that would have the same solution. Consider $10 \div 5$ and $100 \div 50$. These have the same solution because they are proportional, just as $\frac{20}{10} = \frac{10}{5} = \frac{2}{1}$. More generally, $\frac{x}{y} = \frac{10x}{10y}$. Multiplying by 10 is not necessary, any equivalent ratio with a whole number divisor will work for long division.

2.3 Nix: Move the Decimal (Scientific Notation)

Because:



Students get the answer right half the time and do not understand what went wrong the other half of the time. Students are thinking about moving the decimal place, not about place value or multiplying by 10.

Fix:

Students need to ask themselves, "Is this a big number or a small number?" Scientific notation gives us a compact way of writing long numbers, so is this long because it is large or long because it is small? An important idea of scientific notation is that you are not changing the value of the quantity, only its appearance. Writing a number in scientific notation is similar to factoring, but in this case we are only interested in factors of ten.

Consider 6.25 x 10^{-2}.

10^{-2} is the reciprocal of 10^2 so $6.25 \text{ x } 10^{-2} = 6.25 \cdot \frac{1}{10^2} = \frac{6.25}{100} = .0625$

To write 0.0289 in scientific notation, we need to have the first non-zero digit in the ones place, so we need 2.89 x something.

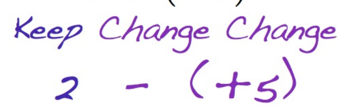
Compare 2.89 with 0.0289:

$$2.89 \text{ x } 10^x = 0.0289 \implies 10^x = \frac{0.0289}{2.89} = \frac{1}{100} \implies x = -2.$$

2.4 Nix: Same-Change-Change, Keep-Change-Change (Integer Addition)

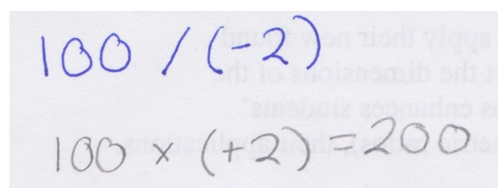
Because:

$$2 + (-5)$$



It has no meaning and there is no need for students to memorize a rule here. They are able to reason about adding integers (and extrapolate to the reals). A trick will circumvent thinking, while a tool like the number line will be a useful strategy throughout their studies.

This student takes Same-Change-Change and expands it to multiplication and division. Because if you can magically switch between addition and subtraction, why not switch multiplication and division?



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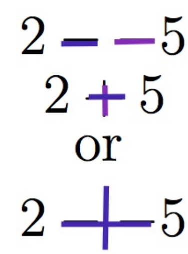
Fix:

Once students are comfortable adding and subtracting whole numbers on the number line, all they need to add to their previous understanding is that a negative number is the opposite of a positive number. Some students may find a vertical number line (or even a representation of a thermometer) more friendly to start with. There are many ways to talk about positive and negative numbers, but shifting on a number line is a good representation for students to be familiar with as the language will reappear during transformations of functions.

$2+5$	Start at 2, move 5 spaces to the *right*
$2+(-5)$	Start at 2, move 5 spaces to the *left* (opposite of right)
$2-5$	Start at 2, move 5 spaces to the *left*
$2-(-5)$	Start at 2, move 5 spaces to the *right* (opposite of left)

2.5 Nix: Two Negatives Make a Positive (Integer Subtraction)

Because:



Again, it has no meaning and there is no need for students to memorize a rule here.

Fix:

As was the case above, students can reason through this on the number line. Once students recognize that addition and subtraction are opposites, and that positive and negative numbers are opposites, they will see that two opposites gets you back to the start. Just as turning around twice returns you to facing forward.

$2-(-5) \Rightarrow$ Start at 2, move (-5) spaces to the *left*

$\Rightarrow$ Start at 2, move 5 spaces to the *right* (opposite of left)

$\Rightarrow 2+5$

Therefore, $2-(-5)=2+5$.

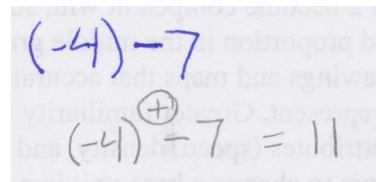
2.6 Nix: Two Negatives Make a Positive (Integer Multiplication)

Because:

$$(-3)(-2) =$$
$$(+3)(+2) = +6$$

Many students will overgeneralize and mistakenly apply this rule to addition as well as multiplication and division.

For example, this student seems to be thinking: "Negative and negative = positive, right?"



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Fix:

Students are able to reason about multiplying integers (and extrapolate to the reals) independently. They can look at patterns and generalize from a few approaches. One option is to use opposites:

We know that $(3)(2) = 6$.

So what should $(-3)(2)$ equal? The opposite of 6, of course!
Therefore, $(-3)(2) = -6$.

So, what should $(-3)(-2)$ equal? The opposite of -6, of course!
Therefore, $(-3)(-2) = 6$.

Another option is to use patterning. Since students are already familiar with the number line extending in both directions, they can continue to skip count their way right past zero into the negative integers.

Students can use the following pattern to determine that $(3)(-2) = -6$:

Product	Result
$(3)(2)$	6
$(3)(1)$	3
$(3)(0)$	0
$(3)(-1)$	-3
$(3)(-2)$	-6

With $(3)(-2) = -6$ in hand, students can use the following pattern to determine that $(-3)(-2) = 6$:

Product	Result
$(3)(-2)$	-6
$(2)(-2)$	-4
$(1)(-2)$	-2
$(0)(-2)$	0
$(-1)(-2)$	2
$(-2)(-2)$	4
$(-3)(-2)$	6

2.7 Nix: PEMDAS, BIDMAS

Because:

Students interpret the acronym in the order the letters are presented, leading them to multiply before dividing and add before subtracting.

P E M D A S
() ^ * / + -

For example, students often incorrectly evaluate $6 \div 2 \cdot 5$ as follows:

Incorrect: $6 \div 2 \cdot 5 = 6 \div 10 = 0.6$

Correct: $6 \div 2 \cdot 5 = \frac{6}{2} \cdot 5 = 3 \cdot 5 = 15$

Fix:

Students should know that mathematicians needed a standard order of operations for consistency. The most powerful operations should be completed first - exponentiation increases or decreases at a greater rate than multiplying, which increases or decreases at a greater rate than addition. Sometimes we want to use a different order, so we use grouping symbols to signify "do this first" when it is not the most powerful operation. If students are still looking for a way to remember the order, replace the confusing acronym PEMDAS with the clearer GEMA.

G is for grouping, which is better than parentheses because it includes all types of groupings such as brackets, absolute value, expressions under square roots, the numerator of a fraction, etc. Grouping also implies more than one item, which eliminates the confusion students experience when they try to "Do the parentheses first." in 4(3).

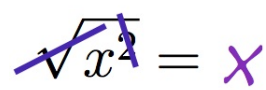
E is for exponents. Nothing new here.

M is for multiplication. Division is implied. Since only one letter appears for both operations, it is essential to emphasize the important inverse relationship between multiplication and division. For example, discuss the equivalence of dividing by a fraction and multiplying by the reciprocal.

A is for addition. Subtraction is implied. Again, since only one letter appears for both operations, it is essential to emphasize the important inverse relationship between addition and subtraction. You might talk about the equivalence of subtraction and adding a negative.

2.8 Nix: The Square Root and the Square Cancel

Because:



Cancel is a vague term, it invokes the image of terms/values/variables magically disappearing. The goal is to make mathematics less about magic and more about reasoning. Something is happening here, let students see what is happening! Plus, the square root is only a *function* when you restrict the domain!

Fix:

Insist that students show each step instead of canceling operations.

$$\sqrt{(-5)^2} \neq -5 \text{ because } \sqrt{(-5)^2} = \sqrt{25} = 5$$

$$x^2 = 25 \quad \not\Rightarrow \quad x = \sqrt{25}$$

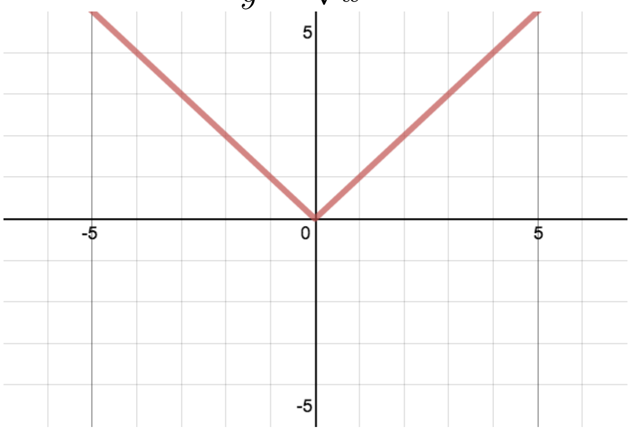
If you "cancel" the square with a square root you miss a solution.

Instead:

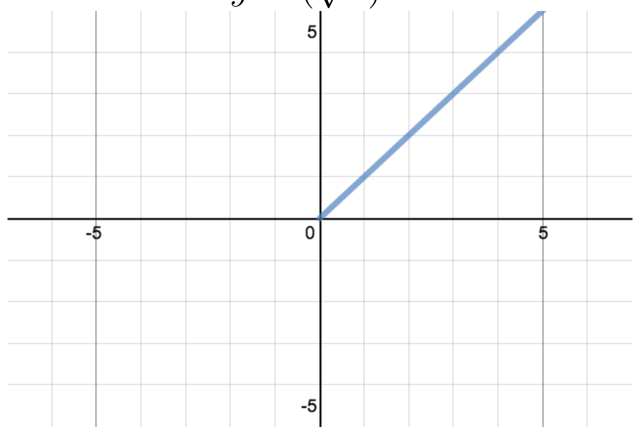
$$\sqrt{x^2} = \sqrt{25} \quad \Leftrightarrow \quad |x| = \sqrt{25} \quad \Leftrightarrow \quad x = \pm\sqrt{25}$$

Another approach is to have students plot $y = \sqrt{x^2}$ and $y = (\sqrt{x})^2$ on their graphing utilities and compare results.

$y = \sqrt{x^2}$



$y = (\sqrt{x})^2$



2.9 Nix: The Log and the Exponent Cancel

Because:

$2^{\log_2 x} = x$

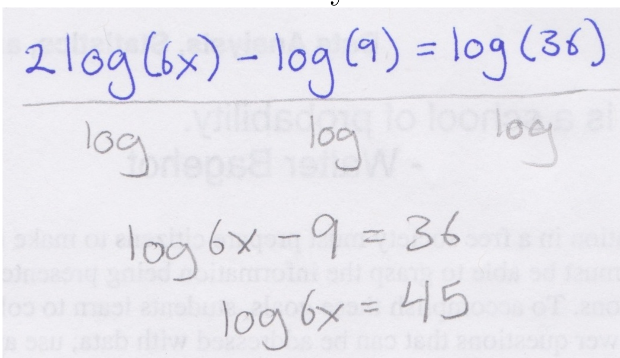
Cancel is a vague term, it invokes the image of terms/values/variables magically disappearing. The goal is to make mathematics less about magic and more about reasoning. Something is happening here, let students see what is happening!

$$\log_2 x = 4$$
$$2^{\log_2 x} = 2^4$$
$$x = 2^4$$

Students think the second line looks scary and it is rather ridiculous to write this step out if you know the definition of a log.

If students are not thinking about the definition of a log, they will try to solve the new function in the ways they are familiar with.

Student divides by the function:



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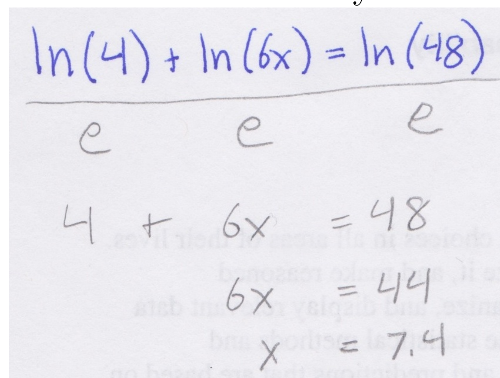
This student divides by the base:



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Fix:

Rewrite the log in exponent form, then solve using familiar methods. This is another reason to teach the definition of logarithms; then there is no confusion about the relationship between logs and exponents.

The log asks the question "*base* to what *power* equals *value*?"

$$\log_{\text{base}} \text{value} = \text{power}$$
$$\text{base}^{\text{power}} = \text{value}$$

$$\log_2 4 = y$$
$$2^y = 4$$
$$y = 2$$

$$\log_2 x = 4$$
$$2^4 = x$$

Chapter 3

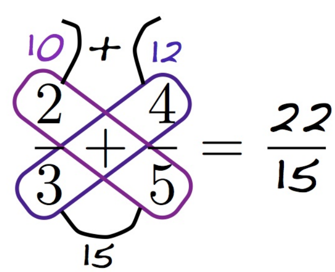
Ratios and Proportional Relationships

Ratios and proportions are a new way of thinking for elementary students. Teachers often bemoan the difficulty that kids have with fractions, but it is because we rob them of the opportunity to develop any intuition with them. The first experience most people have with math is counting, then adding, along with additive patterns. Even when they start multiplying, it tends to be defined as repeated addition. Fractions are the first time when adding will not work, and it messes students up. Skip the shortcuts and let your kids see that fractions, ratios and proportions are multiplicative - a whole new way to interpret the world!

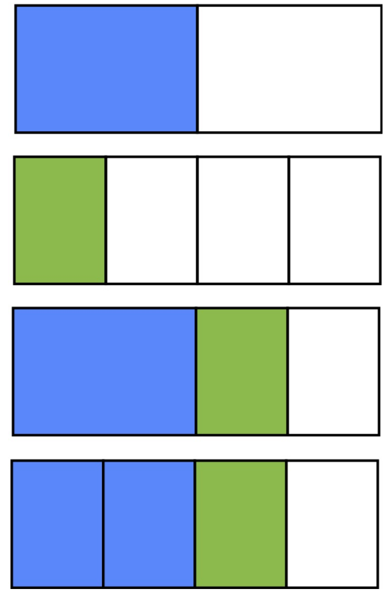
3.1 Nix: Butterfly Method, Jesus Fish

Because:

Students have no idea why it works and there is no mathematical reasoning behind the butterfly, no matter how pretty it is.



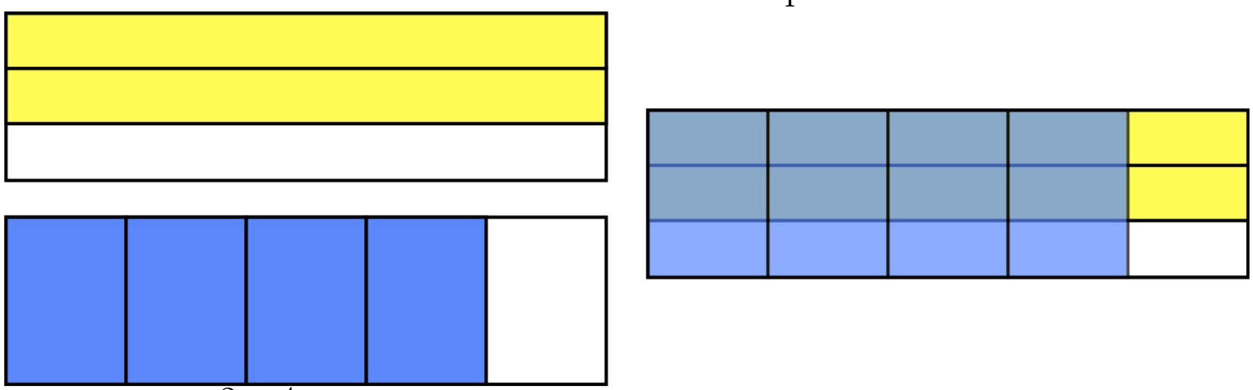
Fix:



If students start with visuals such as fraction strips they will discover the need to have like terms before they can add. Say a student wants to add $\frac{1}{2} + \frac{1}{4}$. They may start with a representation of each fraction, then add the fractions by placing them end to end. The representation is valid, but there is no way to write this new diagram as a single fraction. To do so, students need to cut the whole into equal parts. After some experience, students will realize they need common denominators to add. After still more experience adding fractions with common denominators, students will realize they can simply add the numerators (which is equivalent to counting the number of shaded pieces) while keeping the denominator the same (as the size of the pieces does not change).

Fractions can be compared and added/subtracted with any common denominator, there is no mathematical reason to limit students to the least common denominator. Many visual/manipulative methods will not give least common denominators (instead using the product of the denominators) and that is just fine! Accept any denominator that is computationally accurate. Students may eventually gravitate towards the least common denominator as they look for the easiest numbers to work with. In the meantime, encourage students to compare different methods - do different common denominators give different answers? Are they really different or might they be equivalent? How did that happen? Fractions can even be compared with common numerators - a fascinating discussion to have with students of any age!

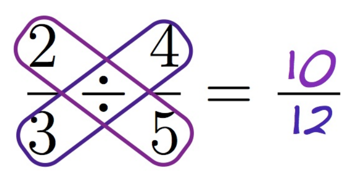
Kids want to use the phrase “Cross Multiply” for everything: How do we multiply fractions? “Cross Multiply!” How do we divide fractions? “Cross Multiply!” How do we solve proportions? “Cross Multiply!” Those are three entirely different processes; they need different names. For multiplication of fractions “Cross Multiply” means “multiply across” (horizontally) and there isn’t usually a trick associated with this operation. I have found that by high school most students don't have any difficulty with this operation - it matches their intuition. In fact, they like this method so much they want to extend it to other operations (non-example: to add fractions, add the numerators and add the denominators) which fails. Instead of saying cross multiply, use the precise (though admittedly cumbersome) phrase "multiply numerator by numerator and denominator by denominator" when students need a reminder of how to multiply fractions. Or better yet, avoid using any phrase at all and direct students to an area model to determine the product.



To multiply $\frac{2}{3} \cdot \frac{4}{5}$ find the area shaded by both - that is two-thirds of four-fifths. The fifths are each divided into thirds and two of the three are shaded. The resulting product is $\frac{8}{15}$.

3.2 Nix: Cross Multiply (Fraction Division)

Because:



Division and multiplication are different (albeit related) operations, one cannot magically switch the operation in an expression. Plus, students confuse "cross" (diagonal) with "across" (horizontal). Not to mention, where does the answer go? Why does one product end up in the numerator and the other in the denominator?

Fix:

Use the phrase "multiply by the reciprocal" but only after students understand where this algorithm comes from. The reciprocal is a precise term that reminds students why we are switching the operation.

$$\frac{2}{3} \div \frac{2}{3} = 1 \qquad \text{easy!}$$
$$\frac{2}{3} \div \frac{1}{3} = 2 \qquad \text{makes sense}$$
$$\frac{4}{5} \div \frac{3}{5} = \frac{4}{3} \qquad \text{not as obvious, but still dividing the numerators}$$
$$\frac{4}{5} \div \frac{1}{2} = \ ? \qquad \text{no idea!}$$

If the last problem looked like the previous examples, it would be easier. So let's rewrite with common denominators:

$$\frac{8}{10} \div \frac{5}{10} = \frac{8}{5} \qquad \text{makes sense}$$

If students are asked to solve enough problems in this manner, they will want to find a shortcut and may recognize the pattern. Show them (or ask them to prove!) why multiplying by the reciprocal works:

$$\begin{aligned}\frac{4}{5} \div \frac{1}{2} &= \frac{4 \cdot 2}{5 \cdot 2} \div \frac{1 \cdot 5}{2 \cdot 5} \\ &= \frac{4 \cdot 2}{1 \cdot 5} \\ &= \frac{4 \cdot 2}{5 \cdot 1} \\ &= \frac{4}{5} \cdot \frac{2}{1}\end{aligned}$$

In this case students discover that multiplying by the reciprocal is the equivalent of getting the common denominator and dividing the numerators. This is not an obvious fact. Students will only reach this realization with repeated practice, but practice getting common denominators is a great thing for them to be doing! More importantly, the student who forgets this generalization can fall back on an understanding of common denominators, while the student who learned a rule after completing this exercise once (or not at all!) will guess at the rule rather than attempt to reason through the problem.

A second approach uses compound fractions. Depending on what experience students have with reciprocals, this might be a more friendly option. It has the added bonus of using a generalizable concept of multiplying by "a convenient form of one" which applies to many topics, including the application of unit conversions. To begin, the division of two fractions can be written as one giant (complex or compound) fraction.

$$
\begin{aligned}
\frac{\frac{4}{5}}{\frac{1}{2}} &= \frac{\frac{4}{5}}{\frac{1}{2}} \cdot 1 \\
&= \frac{\frac{4}{5}}{\frac{1}{2}} \cdot \frac{\frac{2}{1}}{\frac{2}{1}} \\
&= \frac{\frac{4}{5} \cdot \frac{2}{1}}{1} \\
&= \frac{4}{5} \cdot \frac{2}{1}
\end{aligned}
$$

3.3 Nix: Flip and Multiply, Same-Change-Flip

Because:

Division and multiplication are different (albeit related) operations, one cannot magically switch the operation in an expression. Plus, students get confused as to what to "flip."

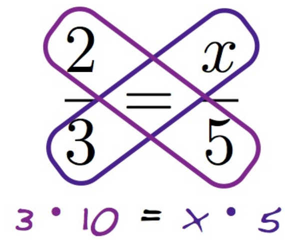
$$\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \cdot \frac{5}{4}$$

Fix:

Use the same methods as described in the fix of Section 3.2.

3.4 Nix: Cross Multiply (Solving Proportions)

Because:



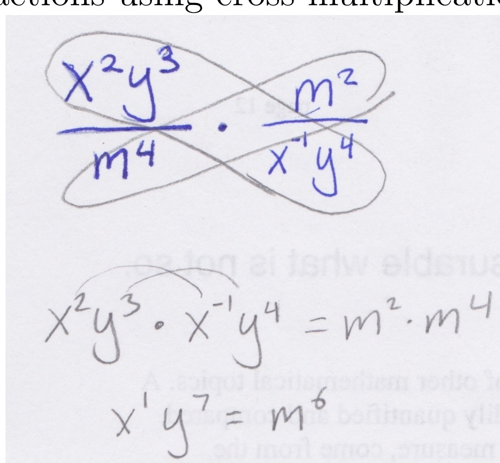
Students confuse "cross" (diagonal) with "across" (horizontal) multiplication, and/or believe it can be use everywhere (such as in multiplication of fractions).

Correct multiplication of fractions: $\frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$

Common error: $\frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 4}{2 \cdot 3} = \frac{4}{6}$

More importantly, you're not magically allowed to multiply two sides of an equation by different values to get an equivalent equation. This process doesn't make sense to students, so they are memorizing a procedure, not understanding a method. Which means that when they forget a step, they guess.

This student tries to multiply fractions using cross multiplication:



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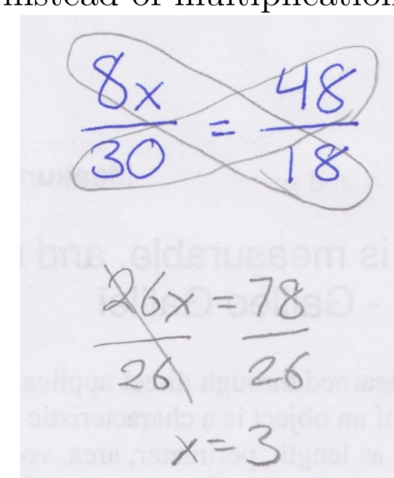
This student uses cross addition instead of multiplication:



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Fix:

Instruct solving all equations (including proportions, they aren't special!) by inverse operations.

$$\begin{aligned} \frac{3}{5} &= \frac{x}{10} \\ 10 \cdot \frac{3}{5} &= 10 \cdot \frac{x}{10} \\ 10 \cdot \frac{3}{5} &= x \\ 6 &= x \end{aligned}$$

Encourage students to look for shortcuts such as common denominators, common numerators or scale factors. Once students know when and why a shortcut works, skipping a few steps is okay, but students must know why their shortcut is "legal algebra" and have a universal method to fall back on.

Shortcuts:

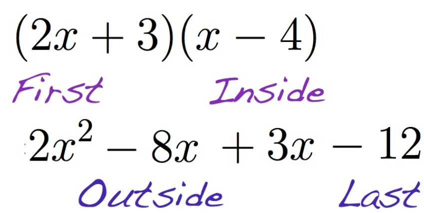
$\frac{3}{5} = \frac{x}{5} \Leftrightarrow x = 3$	no work required, meaning of equal
$\frac{3}{5} = \frac{x}{10} \Leftrightarrow \frac{3 \cdot 2}{5 \cdot 2} = \frac{x}{10} \Leftrightarrow x = 6$	multiply a fraction by 1 to get an equivalent fraction
$\frac{4}{8} = \frac{x}{10} \Leftrightarrow x = 5$	students are quick to recognize $\frac{1}{2}$ but any multiplicative relationship works
$\frac{5}{3} = \frac{10}{x} \Leftrightarrow \frac{3}{5} = \frac{x}{10}$	take the reciprocal of both sides of the equation to get the variable into the numerator

Chapter 4

Arithmetic With Polynomials

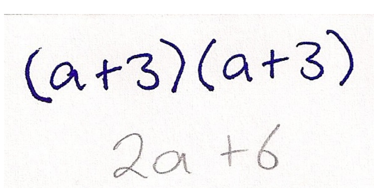
4.1 Nix: FOIL

Because:



It implies an order - a few of my Honors Pre-Calculus students were shocked to learn that OLIF works just as well as FOIL does. It is also a one trick pony. There are other ways to multiply binomials that are transferrable to later work such as multiplying larger polynomials and factoring by grouping.

When students memorize a rule without understanding they will misapply it.



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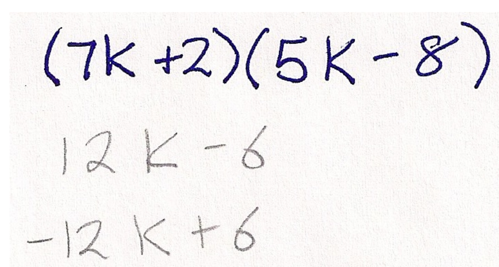

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Fix:

Replace FOIL with the distributive property. It can be taught as soon as distribution is introduced. Students can start by distributing one binomial to each part of the other binomial. Then distribution is repeated on each monomial being multiplied by a binomial. As students repeat the procedure they will realize that each term in the first polynomial must be multiplied by each term in the second polynomial. This pattern, which you might term "each by each" carries through the more advanced versions of this exercise.

In elementary school students learn an array model for multiplying number. The box method builds on this knowledge of partial products.

$$\begin{aligned} 23 \cdot 45 =& (20+3)(40+5) \\ =& 20(40+5) + 3(40+5) \\ =& 20 \cdot 40 + 20 \cdot 5 + 3 \cdot 40 + 3 \cdot 5 \\ =& 800 + 100 + 120 + 15 \end{aligned}$$

	40	**5**
20	800	100
3	120	15

$$\begin{aligned} & (2x+3)(x-4) \\ =& (2x+3)(x) + (2x+3)(-4) \\ =& 2x^2 + 3x - 8x - 12 \\ =& 2x^2 - 5x - 12 \end{aligned}$$

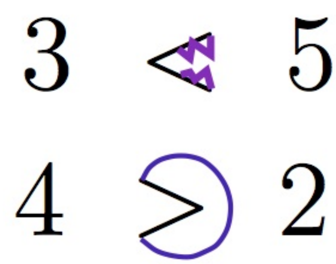
	2x	**3**
x	$2x^2$	$3x$
−4	$-8x$	-12

Chapter 5

Reasoning with Equations and Inequalities

5.1 Nix: 'Hungry' Inequality Symbols

Because:



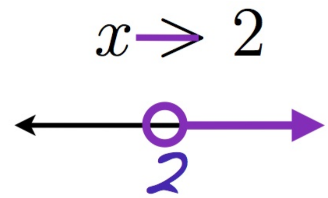
Students get confused with the alligator/pacman analogy. Is the bigger value eating the smaller one? Is it the value it already ate or the one it is about to eat?

Fix:

Ideally students have enough exposure to these symbols that they memorize the meaning. Just as they see and write $=$ when they hear or say "equal," students should see and write $<$ when they hear or say "less than." To help students before they internalize the meaning, have them analyze the shape. Instead of the segments being parallel like in an equal symbol, which has marks that are the same distance apart on both sides, the bars have been tilted to make a smaller side and a larger side. The greater number is next to the wider end and the lesser number is next to the narrower end. Beware of language here: use "greater" rather than "bigger" because when integers are brought into play "bigger" creates trouble.

5.2 Nix: Follow the Arrow (Graphing Inequalities)

Because:



The inequalities $x > 2$ and $2 < x$ are equivalent and equally valid, yet they point in opposite directions. Plus, not all inequalities will be graphed on a horizontal number line!

Fix:

Students need to understand what the inequality symbol means. Ask students, "Are the solutions greater or lesser than the endpoint?" This is a great time to introduce test points - have students plot the endpoint, test a point in the inequality and then shade in the appropriate direction. While this seems like more work than knowing which direction to shade, it is a skill that applies throughout mathematics (including calculus!).

Since the symbolic representation of an inequality is more abstract than a number line, having students practice going from context or visual representations to symbolic ones will support student understanding of the symbols. While it is true that $x > 2$ is the more natural way to represent the sentence "the solutions are greater than two," students need the versatility of reading inequalities in both directions for compound inequalities. For example, $0 < x < 2$ requires students to consider both $0 < x$ and $x < 2$.

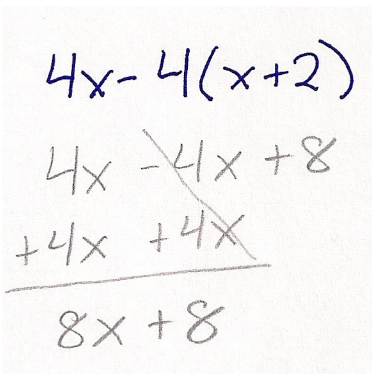
5.3 Nix: Cancel

$$\frac{\cancel{5}x}{\cancel{5}} = 10$$

$$x = 10$$

Because:

Cancel is a vague term that hides the actual mathematical operations being used, so students do not know when or why to use it. To many students, cancel is digested as "cross-out stuff" by magic, so they see no problem with crossing out parts of an expression or across addition.



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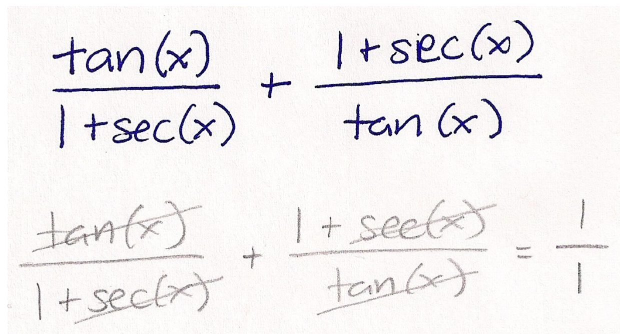

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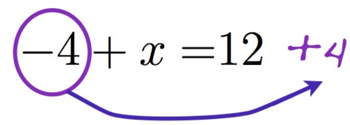
Fix:

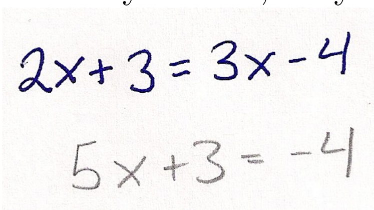
Instead of saying cancel, require students to state a mathematical operation or description along with those lines drawn through things that "cancel." In fractions, we are dividing to get 1. Students can say "divides to one" and show that on their paper by making a big 1 instead of a slash to cross things out or circling the terms and writing a 1 next to them. Emphasizing the division helps students see that they cannot cancel over addition (when students try to cross out part of the numerator with part of the denominator, for example). On opposite sides of an equation, we are subtracting the same quantity from both sides or adding the opposite to both sides. Students can say "adds to zero" and show that on their paper by circling the terms and thinking of the circle as a zero or writing a 0 next to them. In general, use the language of inverse operations, opposites and identities to precisely define the mysterious "cancel."

5.4 Nix: Take/Move to the Other Side

Because:

Taking and moving are not algebraic operations. There are mathematical terms for what you are doing, use them! When students think they can move things for any reason, they will neglect to use opposite operations.





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Fix:

Using mathematical operations and properties to describe what we are doing will help students develop more precise language. Start solving equations using the utmost precision: "We add the opposite to both sides. That gives us zero on the left and leaves the +4 on the right."

$$\begin{aligned} -4 + x &= 12 \\ +4 \qquad &\quad + 4 \\ \hline 0 + x &= 12 + 4 \end{aligned}$$

Similarly for multiplication and division, demonstrate: "If I divide both sides by three, that will give one on the right and a three in the denominator on the left."

$$\frac{-4}{3} = \frac{3x}{3}$$

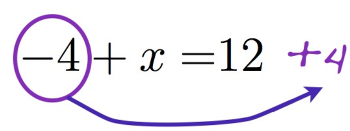
$$\frac{4}{3} = 1x$$

After time, students will be able to do some steps mentally and will omit the zero and the one in their written work, but they must be able to explain why the numbers disappeared. This is essential as when they reach more complex problem solving the identities may need to remain for the equation to make sense. For example, the one might be all that remains of the numerator of a fraction - the denominator must be under *something*!

5.5 Nix: Switch the Side and Switch the Sign

Because:

This ditty hides the actual operation being used. Students who memorize a rhyme have no idea what they are doing. This leads to misapplication and the inability to generalize appropriately.



Fix:

Talk about inverse operations or getting to zero (in the case of addition) or one (in the case of multiplication) instead. The big idea is to maintain the equality by doing the same operation to both quantities.

Chapter 6

Interpreting Functions

6.1 Nix: What is b?

Because:

$$y = mx + b$$

The answer to this question is, "b is a letter." It does not have any inherent meaning, instead ask students exactly what you are looking for, which is probably either the y-intercept or a point to use when graphing the line.

Fix:

There are many equations of a line, $y = mx + b$ is just one of them. Given the goal of graphing, feel free to ask what the intercept is, but it is much easier to ask for any point. Have students pick a value for x (eventually they will gravitate toward zero anyway since it is easy to multiply by!) and then solve for y. This method works for any equation and it shows why we put the value of b on the y-axis. Remind students that, while b is a number, we are concerned with $(0, b)$ which is a point.

Chapter 7

Geometry

In Geometry the lack of concept development comes from a different angle - formulas without background. These are not tricks, but if they are taught without context they become as magical as any other trick in this book. Formulas can be discovered, explored, derived or understood depending on the course, but if they are merely memorized they are no better than a meaningless shortcut. Giving students a formula without letting them experience the process takes away the thinking involved.

7.1 Formula: Distance Formula

Understanding:

Please do not teach the distance formula before the Pythagorean Theorem! It is much harder for students to remember and does not provide any additional meaning. In my everyday life I only use the Pythagorean Theorem, nothing more. And as a math teacher, finding the distance between points is something I do somewhat regularly. Manipulating the Pythagorean Theorem to solve for different variables gives students practice solving equations. That said, it does not hurt students to see the distance formula somewhere in their mathematical career, so long as they see how it comes directly from the Pythagorean Theorem. The Pythagorean Theorem also gives us the equation for a circle, that one small formula turns out to be very powerful.

7.2 Formula: Area of Quadrilaterals, Triangles

Understanding:

Start with an understanding that $A = bh$ for rectangles. Students have seen arrays and may immediately make the connection to multiplication; for the other students, have them begin with rectangles having integer side lengths and count boxes. They will want a shortcut and can figure out that multiplying the side lengths will give the area. While length and width are perfectly acceptable names for the sides, base and height are more consistent with other shapes.

Next explore how a parallelogram has the same area as a rectangle with equal base and height. Have students try some examples, then use decomposition to show that a right triangle can be cut from one end of the parallelogram and fit neatly onto the other end creating a rectangle. Thus, $A = bh$ for this shape as well!

Now is the time to look at triangles. I am always tempted (as are most students) to relate the area of a triangle to the area of a rectangle, but that connection only applies to right triangles. Allow students to see that *any* triangle is half of a parallelogram, thus: $A = \frac{1}{2}bh$.

And once they know the formula for a triangle, students can find the area of any two-dimensional figure by decomposition. Depending on the level, students could generalize their decompositions to find a simplified formula for familiar shapes such as trapezoids, but memorization of such formulas is unnecessary.

7.3 Formula: Surface Area

Understanding:

Many students do not realize that surface area is the sum of the areas of the faces of a shape. They may have been given a formula sheet with a box labeled Lateral Surface Area and another labeled Total Surface Area and it is a matching game:

1. Find the key words in the question.
2. Match those to the words in the formula sheet.
3. Substitute numbers until the answer matches one of the choices provided.

Instead of playing the game, define surface area as the area of the surfaces. If students can identify the parts of the object (and have mastered 2-D area), then they can find the surface area. As a challenge task, ask students to draw 2-D nets that will fold into the object they are presented with. This is a practice in visualizing complex objects and understanding their connections to simpler objects. That sounds a lot like "Look for and make use of structure" a.k.a. Standard for Mathematical Practice 7 from the Common Core State Standards. The task of breaking down a cone or cylinder into its parts is one that is particularly difficult for students to do. Giving students paper nets to be able to alternately build and flatten is advisable.

Depending on the level, students could generalize their process to find a simplified formula for familiar shapes such as rectangular prisms, but memorization of such formulas is unnecessary.

7.4 Formula: Volume

Understanding:

Just as surface area is the application of familiar area formulas to a new shape, volume formulas are not completely different for every shape, in fact they continue to apply familiar area formulas. There are two basic volume formulas: the prism and the pyramid. The prism formula is rather intuitive if student imagine stacking objects (or physically stack objects such as pattern blocks) they will realize that volume of a prism is the area of the base multiplied by the height. Once again, finding the value in question becomes a practice in visualizing - What is the base? How do I find its area? What is the height?

The formula for a pyramid is not as intuitive, but students should recognize that the area of a pyramid is less than the area of a prism with the same base and height. Some experimentation should show that the correct coefficient is

$\frac{1}{3}$. This covers all shapes traditionally taught through high school (including cones as circular pyramids).

Appendix A

Index of Tricks by Standards

We used the Common Core State Standards.

3.OA.B.5	Section 2.1 Add a Zero (Multiplying by 10)
4.MD.A.3	Section 7.2 Area of Quadrilaterals, Triangles
4.NBT.A.2	Section 5.1 'Hungry' Inequality Symbols
5.MD.C.5	Section 7.2 Area of Quadrilaterals, Triangles
5.OA.A.1	Section 2.7 PEMDAS, BIDMAS
5.NBT.B.7	Section 2.2 Ball to the Wall
5.NF.A.1	Section 3.1 Butterfly Method, Jesus Fish
6.EE.A.1	Section 2.7 PEMDAS, BIDMAS
6.EE.B.7	Section 5.4 Take/Move to the Other Side
6.EE.B.7	Section 5.5 Switch the Side, Switch the Sign
6.G.A.1	Section 7.2 Area of Quadrilaterals, Triangles
6.NS.A.1	Section 3.2 Cross Multiply (Fraction Division)
6.NS.A.1	Section 3.3 Flip and Multiply, Same-Change-Flip
6.NS.B.4	Section 4.1 FOIL

7.EE.B.4	Section 5.2 Follow the Arrow (Graphing Inequalities)
7.EE.B.4	Section 5.4 Take/Move to the Other Side
7.EE.B.4	Section 5.5 Switch the Side, Switch the Sign
7.G.B.6	Section 7.3 Surface Area
7.G.B.6	Section 7.4 Volume
7.NS.A.1	Section 2.4 Same-Change-Change, Keep-Change-Change
7.NS.A.1	Section 2.5 Two Negatives Make a Positive (Integer Subtraction)
7.NS.A.2	Section 2.6 Two Negatives Make a Positive (Integer Multiplication)
7.RP.A.3	Section 3.4 Cross Multiply (Solving Proportions)
8.EE.A.2	Section 2.8 The Square Root and the Square Cancel
8.EE.A.4	Section 2.3 Move the Decimal (Scientific Notation)
8.EE.C.7	Section 5.4 Take/Move to the Other Side
8.EE.C.7	Section 5.5 Switch the Side, Switch the Sign
8.F.A.3	Section 6.1 What is b?
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A-REI.A.1	Section 5.4 Take/Move to the Other Side
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F-BF.B.5	Section 2.9 The Log and the Exponent Cancel
G-GPE.B.7	Section 7.1 Distance Formula
SMP 6	Section 5.3 Cancel
SMP 6	Section 5.4 Take/Move to the Other Side

Appendix B

Types of Tricks

B.1 Tricks Students Misinterpret

Section 2.7 PEMDAS, BIDMAS
Section 5.1 'Hungry' Inequality Symbols
Section 5.2 Follow the Arrow (Graphing Inequalities)

B.2 Methods Eliminating Options

Section 3.4 Cross Multiply (Solving Proportions)
Section 4.1 FOIL (Binomial Multiplication)
Section 6.1 What is b?
Section 7.1 Distance Formula

B.3 Math as Magic, Not Logic

Section 2.1 Add a Zero (Multiplying by 10)
Section 2.2 Ball to the Wall
Section 2.3 Move the Decimal
Section 2.4 Same-Change-Change or Keep-Change-Change
Section 2.5 Two Negatives Make a Positive (Integer Addition)
Section 2.6 Two Negatives Make a Positive (Integer Multiplication)
Section 3.1 Butterfly Method, Jesus Fish
Section 3.2 Cross Multiply (Fraction Division)

Section 3.3 Flip-And-Multiply, Same-Change-Flip
Section 5.4 Take/Move to the Other Side
Section 5.5 Switch the Side and Switch the Sign

B.4 Imprecise Language

Section 2.8 The Square Root and the Square Cancel
Section 2.9 The Log and the Exponent Cancel
Section 5.3 Cancel

Appendix C

Back Cover

We are always adding new tricks, head to NixTheTricks.com to check out the ones currently in the commentary stage or submit a trick you hate to see.

"Every time my students do a trick they have learned somewhere, I say what they are *really* doing out loud."
-Julie Reulbach

"The worst thing about mnemonics is not that they almost always fall apart, they don't encourage understanding, and never justify anything; it's that they kill curiosity and creativity - two important character traits that too many math teachers out there disregard."
-Andy Martinson