EXPLORING NUMBERS IN CONTEXT

How many?

Days in the year = 365

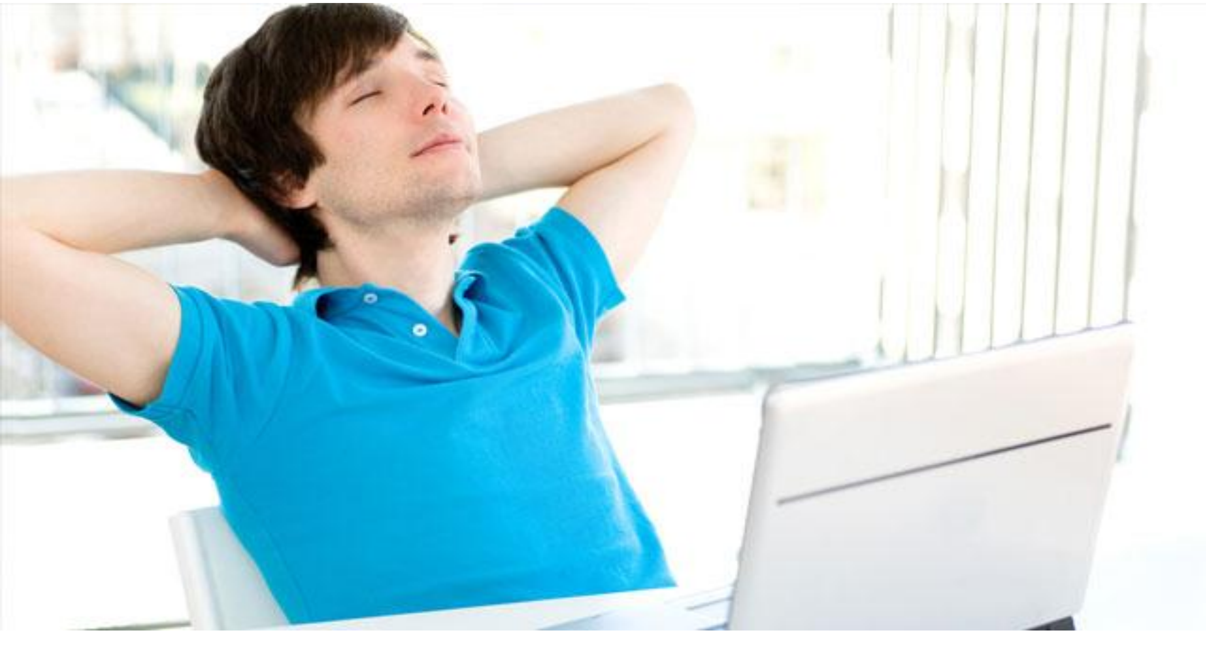
Hours in the year?

Minutes in the year?

Seconds in the year?

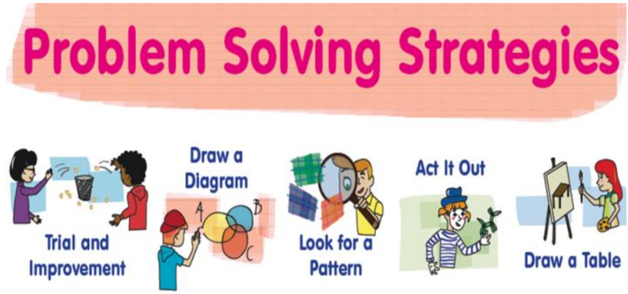
I have €1 to invest for a year and a bank offered me a whopping 100% interest?

What do I expect to get?



Flip chart: Define variables

I have €1 to invest for a year at100% interest?



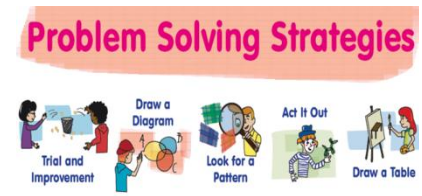
1

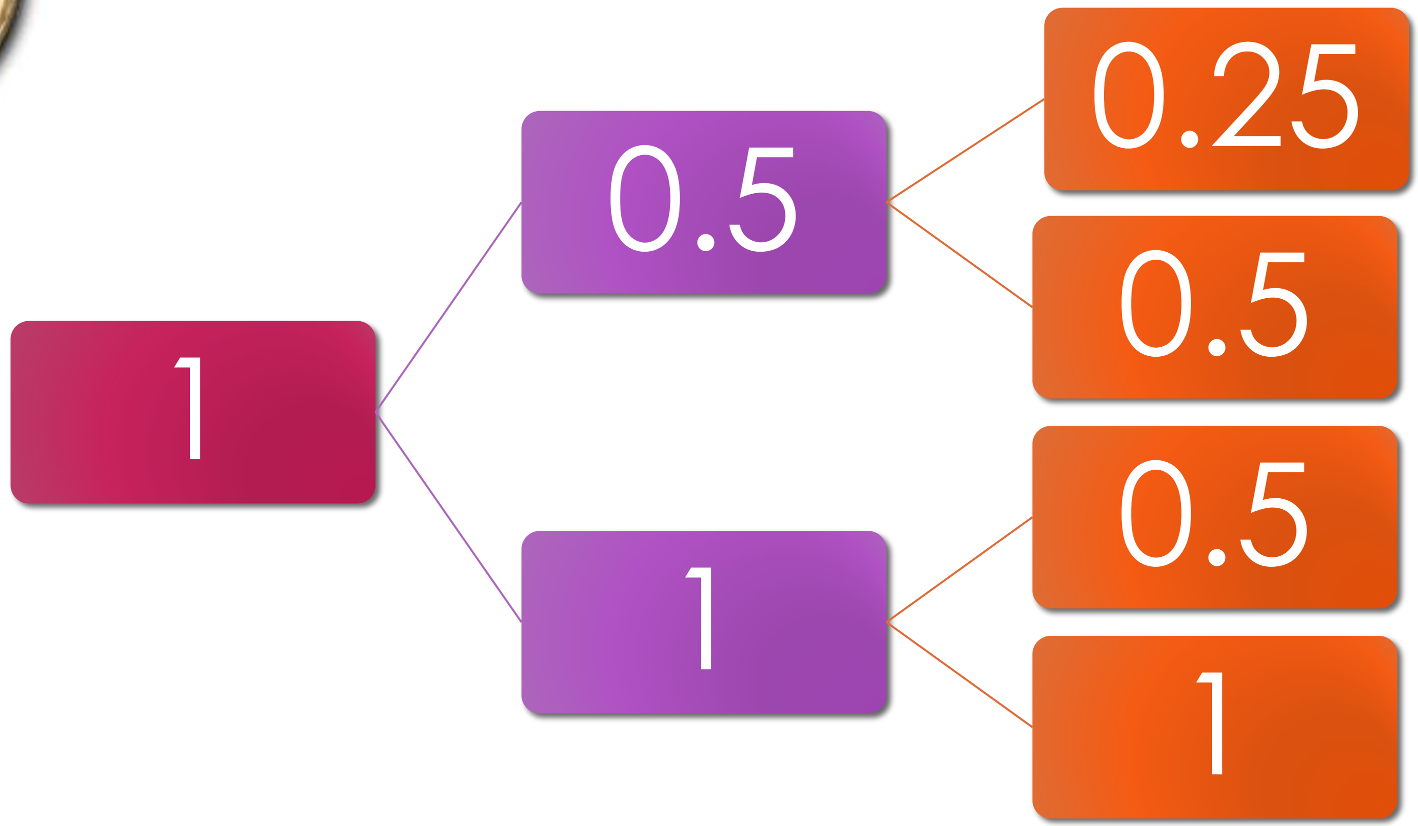
1

1

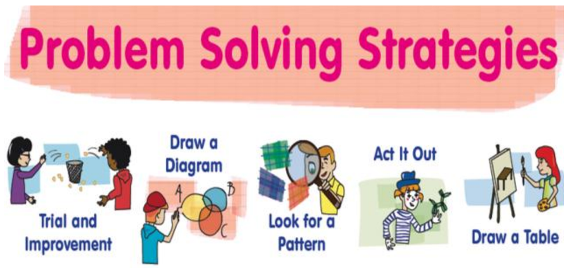
$$1(1+1) = 1 + 1 = 2$$

Every six months?

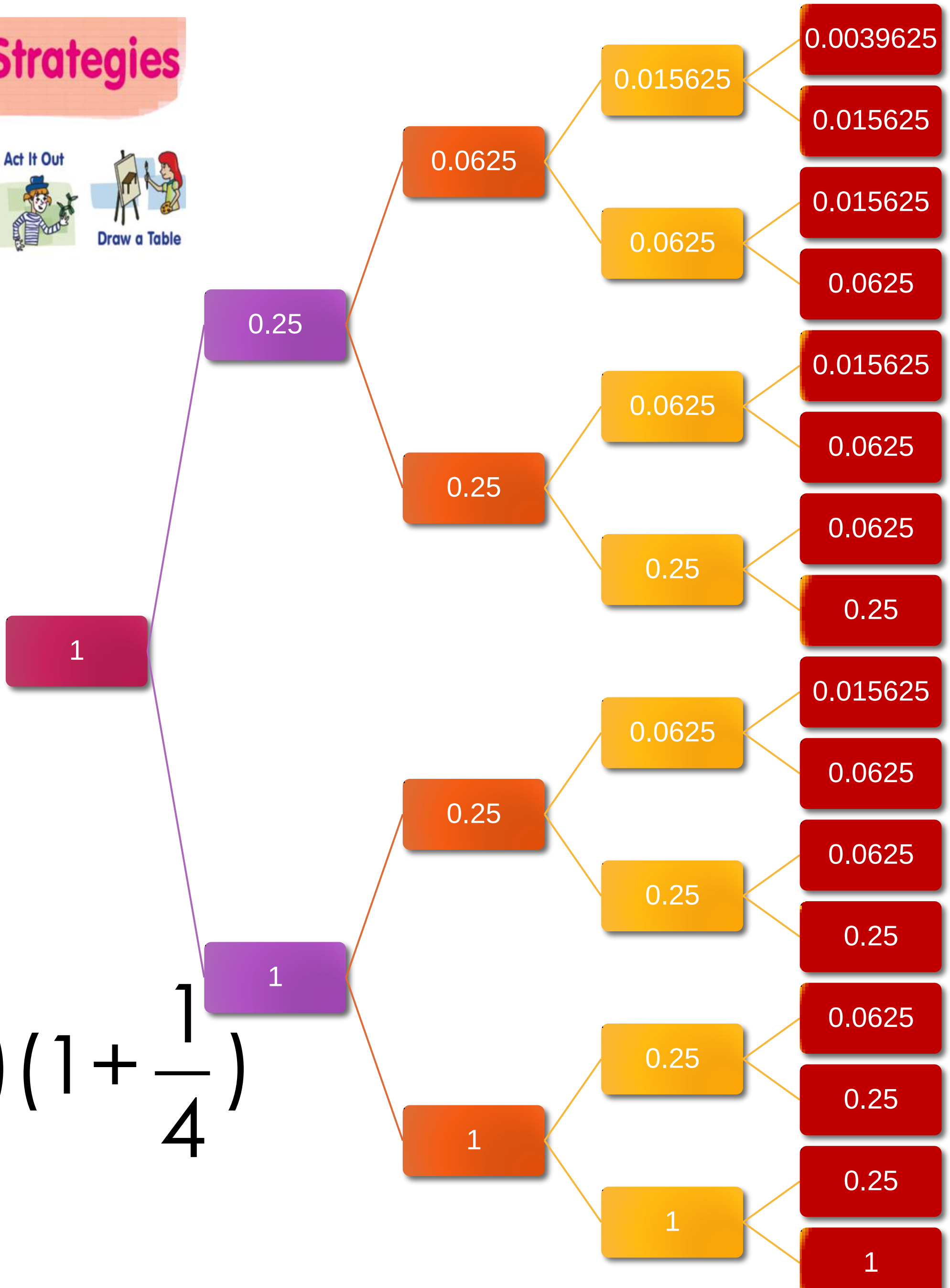




$$1\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)=1+0.5+0.5+0.25=2.25$$



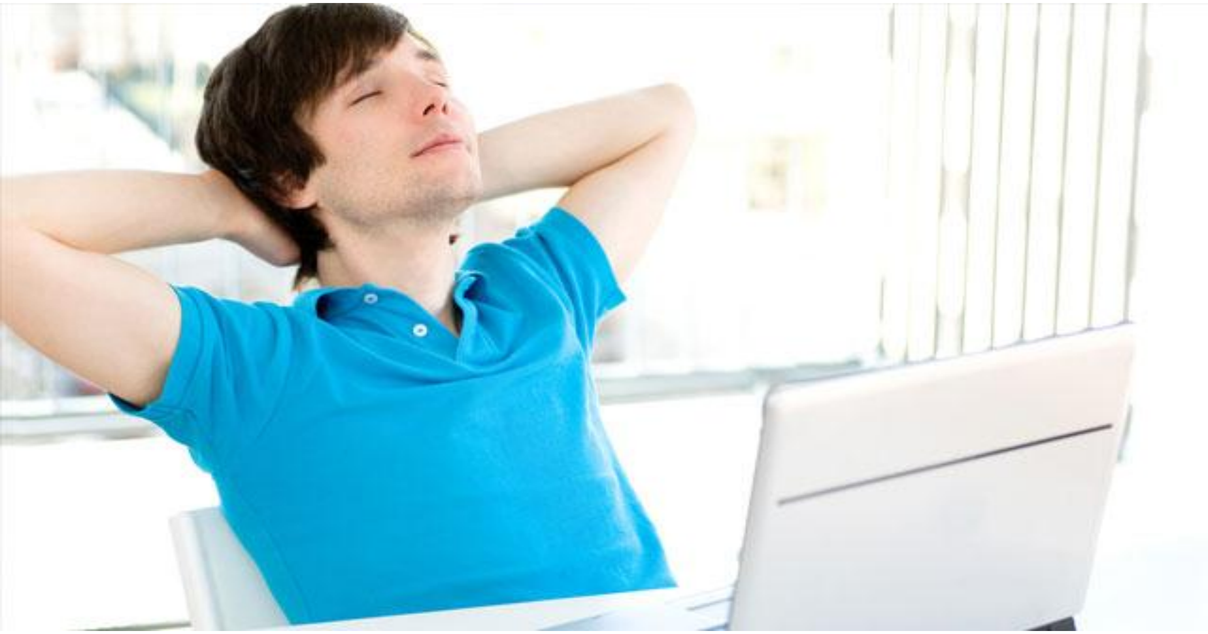
Every 3 months?



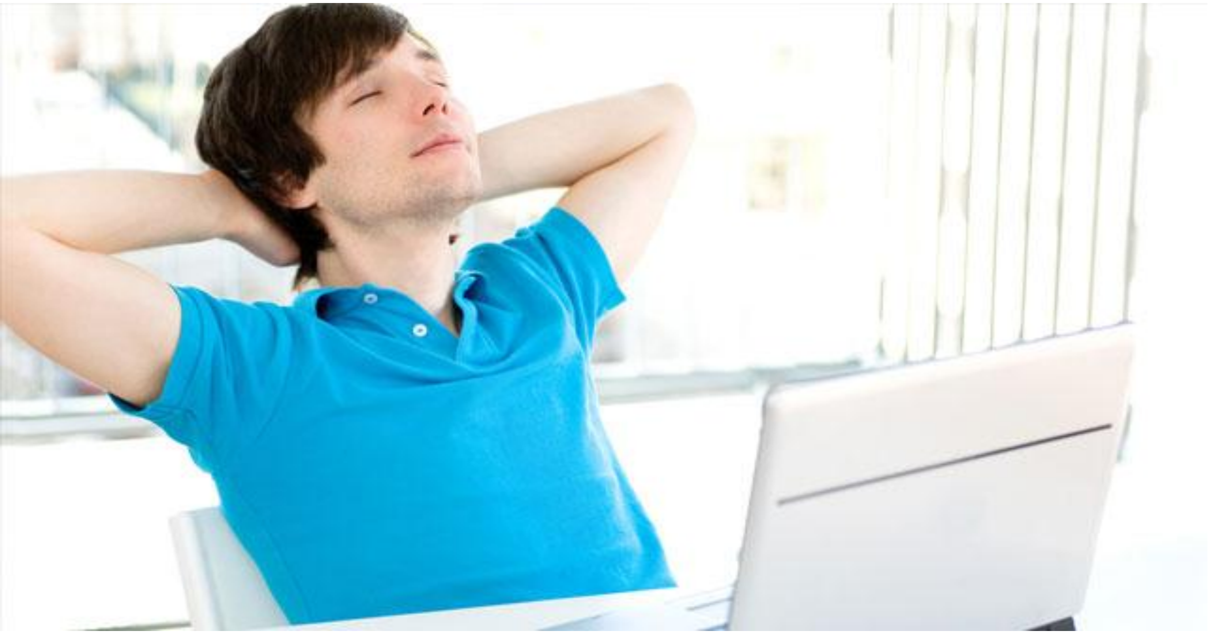
$$1\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)$$

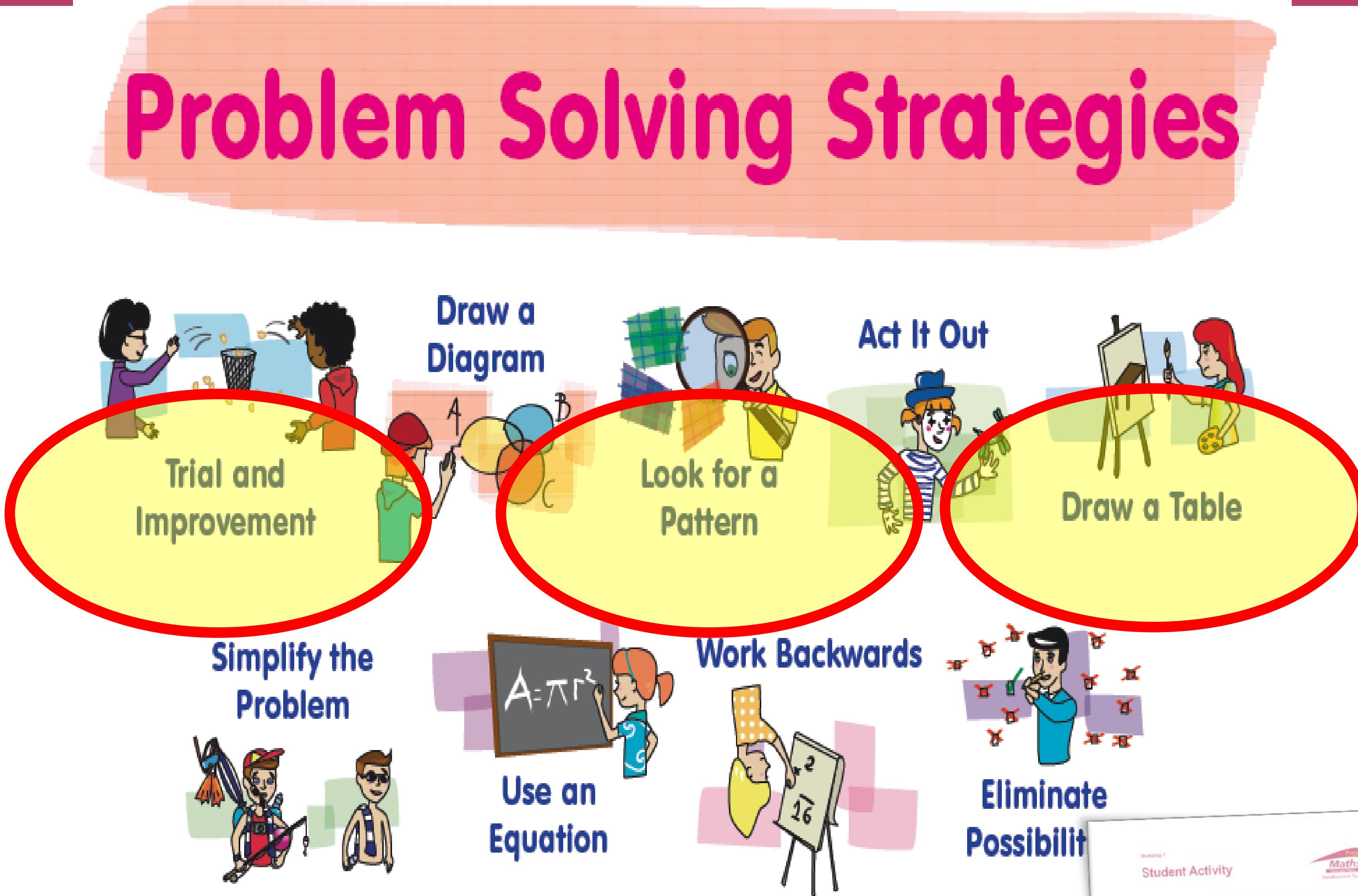
$=2.44140625$

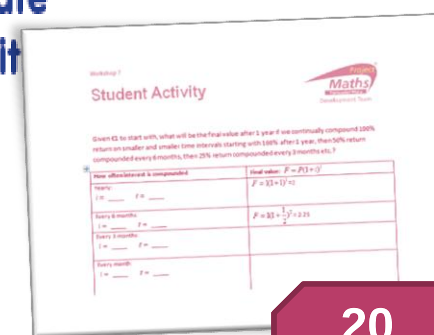
My money was compounded
Every six months?
Every three months?



My money was compounded
Every six months?
Every three months?
Every month?
Every day?
Every hour………




Problem Solving Strategies
Draw a Diagram
Act It Out
Trial and Improvement
Look for a Pattern
Draw a Table
Simplify the Problem
Work Backwards
Use an Equation
Eliminate Possibilit


Student Activity

How often interest is compounded	Final Value F	
Yearly	$F = 1(1+1)^{1}$	= 2.0
Every 6 months	$F = 1\left(1+\frac{1}{2}\right)^{2}$	= 2.25
Every 3 months	$F = 1\left(1+\frac{1}{4}\right)^{4}$	= 2.44140625
Every month	$F = 1\left(1+\frac{1}{12}\right)^{12}$	= 2.61303529
Every week	$F = 1\left(1+\frac{1}{52}\right)^{52}$	= 2.69259695
Every day	$F = 1\left(1+\frac{1}{365}\right)^{365}$	= 2.71456748
Every hour	$F = 1\left(1+\frac{1}{365(24)}\right)^{365(24)}$	= 2.71812669
Every minute	$F = 1\left(1+\frac{1}{365(24)(60)}\right)^{365(24)(60)}$	= 2.71827923
Every second	$F = 1\left(1+\frac{1}{365(24)(60)(60)}\right)^{365(24)(60)(60)}$	= 2.71828162

Let's generalise……. If we divide the year into n compounding periods, how do we find F?

Yearly	$F = 1(1+1)^2$	= 2.0
Every 6 months	$F = 1\left(1+\frac{1}{2}\right)^2$	= 2.25
Every 3 months	$F = 1\left(1+\frac{1}{4}\right)^4$	= 2.44140625
Every month		2.61303529
Every week		2.69259695
Every day	$F = 1\left(1+\frac{}{365}\right)$	= 2.71456748
Every hour	$F = 1\left(1+\frac{1}{8760}\right)^{8760}$	= 2.71812669
Every minute	$F = 1\left(1+\frac{1}{525600}\right)^{525600}$	= 2.71827923
Every second	$F = 1\left(1+\frac{1}{31536000}\right)^{31536000}$	= 2.71828162

$$F = 1\left(1+\frac{1}{n}\right)^n$$

Compounding continued for even smaller compounding periods?

It looks as if any further increases in number of compounding periods will hardly affect the outcome – **the changes will occur in less and less significant digits**.

2.7
2.71
2.718
2.7182
2.71828
2.718281
2.7182818
2.71828182
2.718281828
2.7182818284..............

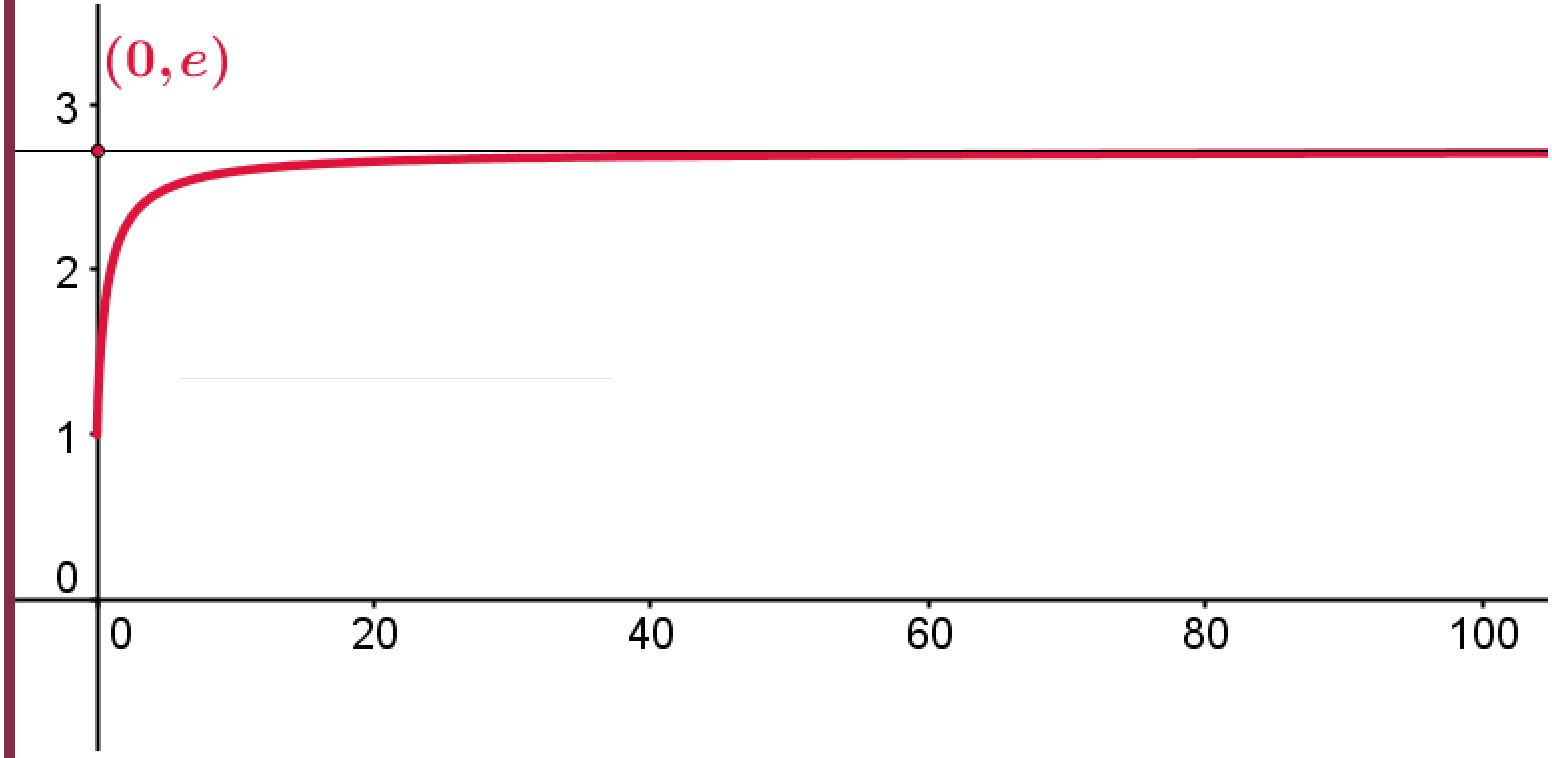
$$F = \left(1 + \frac{1}{n}\right)^{n}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n} = e$$

calculator

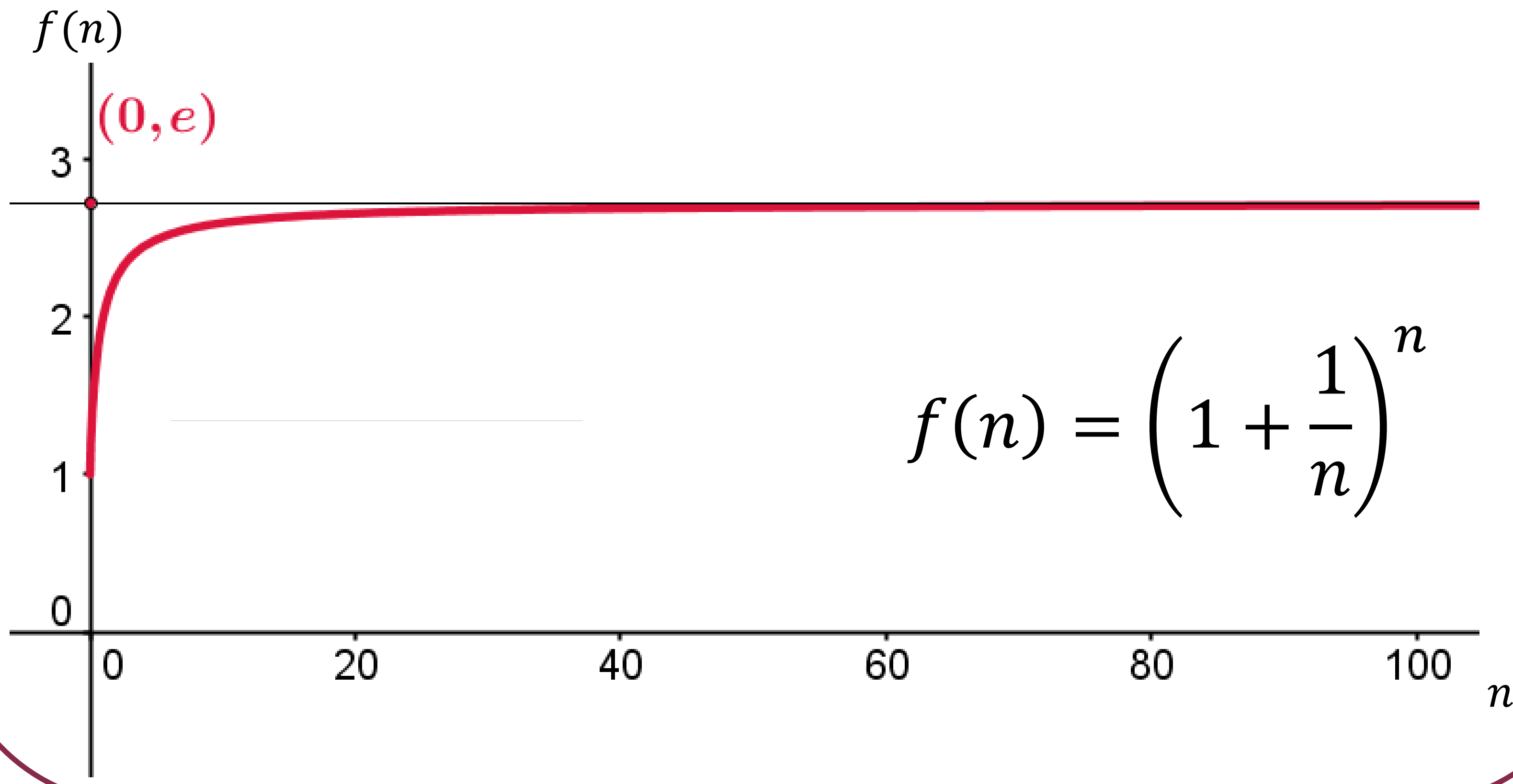
If we start with €1 and compound continuously for 1 year at 100% we get € e.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



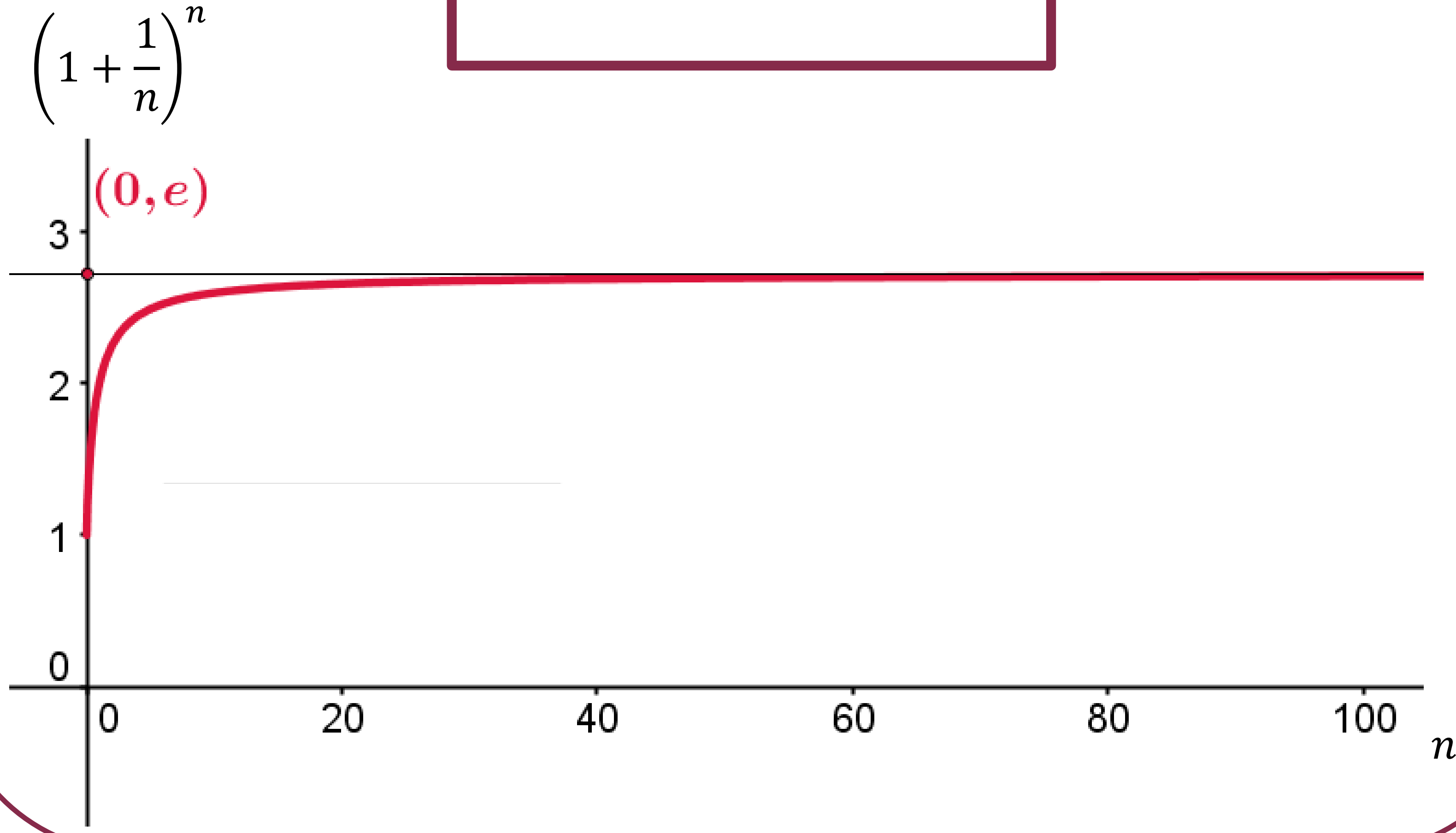
If we start with €1 and compound continuously for 1 year at 100% we get €e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



If we start with €1 and compound continuously for 1 year at 100% we get € e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



If we start with €1 and compound continuously for 1 year at 100% we get €e

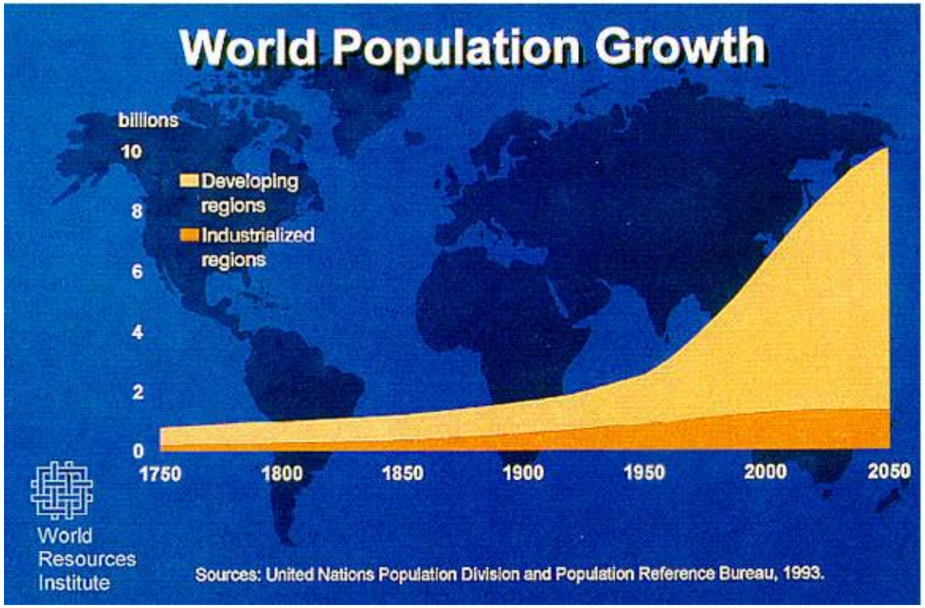
e is not **just** a number………

- ***e* is the base rate of growth shared by all continually growing processes.**

- ***e* shows up whenever systems grow exponentially and continuously:** population, radioactive decay, interest calculations, and more.

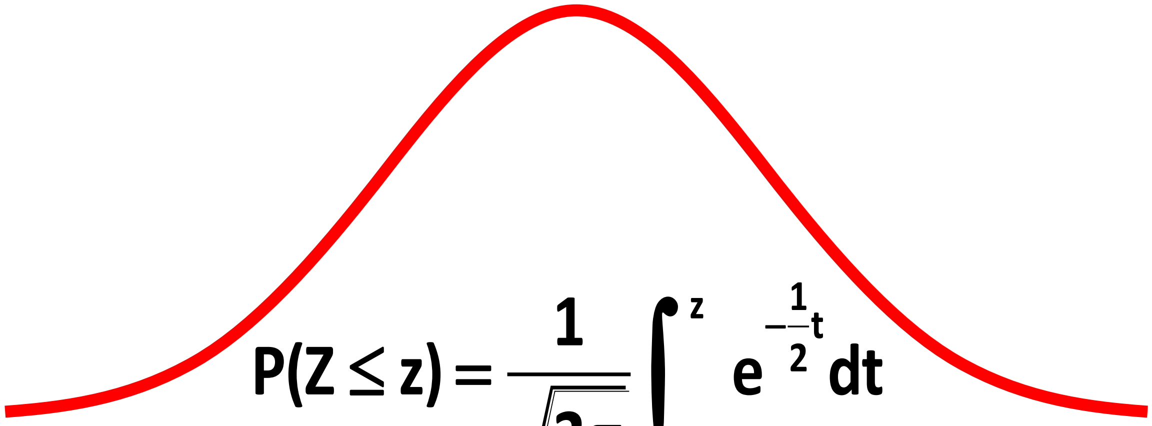
- **Even jagged systems that don’t grow smoothly can be *approximated* by *e*.**

e



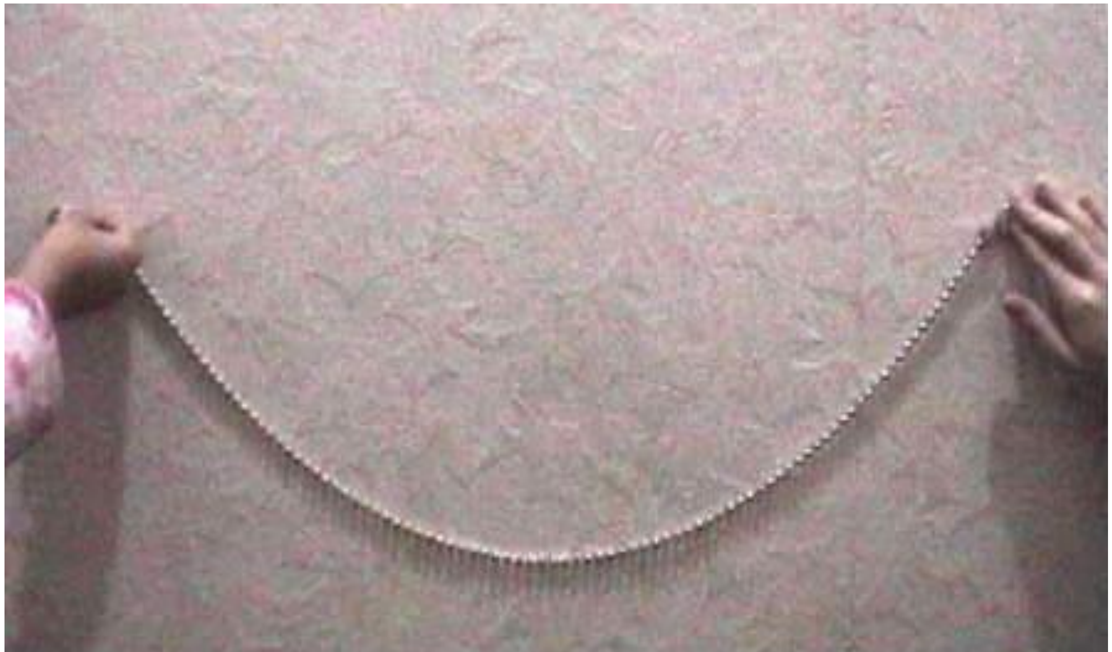
$$N = N_0 e^{-\lambda t}$$

e



$$P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t} dt$$

Where we find e…….



$$y = \frac{e^{ax} + e^{-ax}}{2a}$$

where a is a constant

First 10-digit prime found in consecutive digits of e

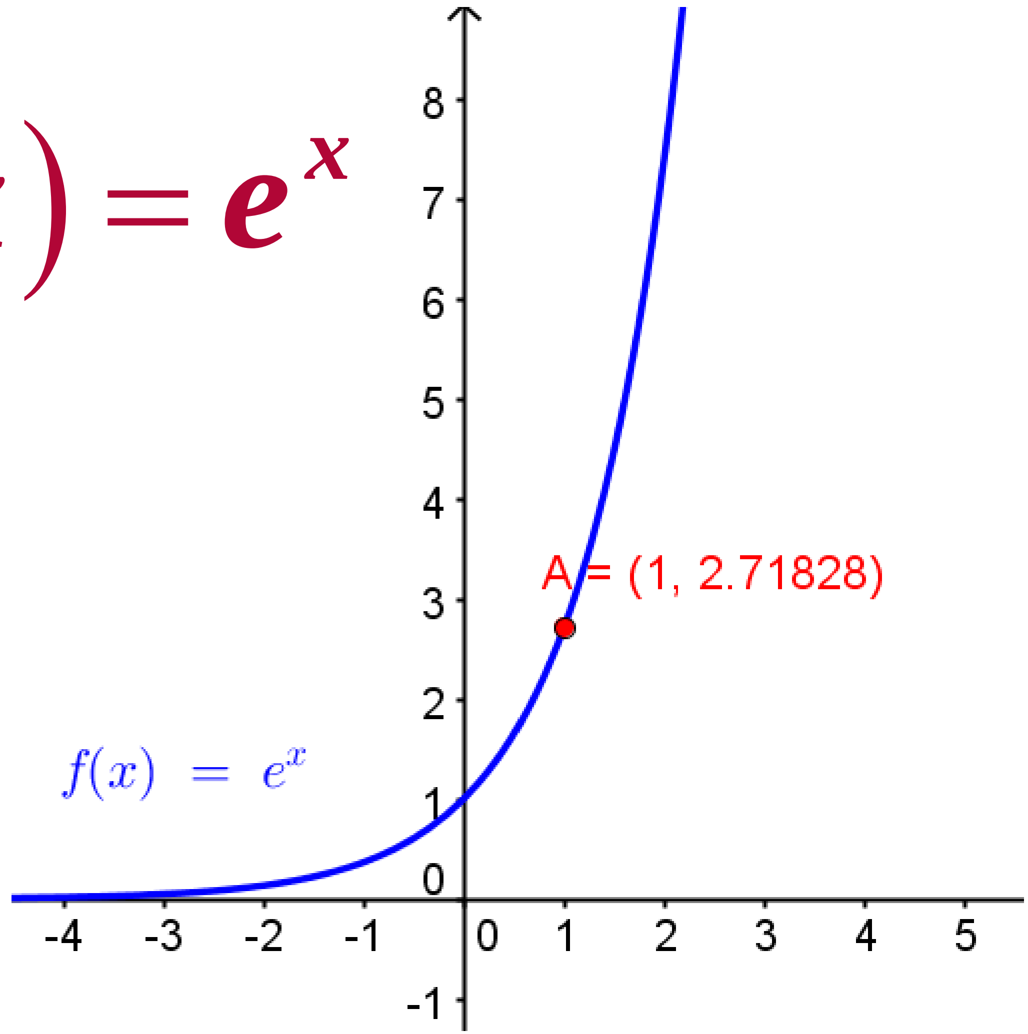
The billboard shown above is part of a **recruiting campaing of Google**. I am far from smart enough to solve this problem, and it this moment I haven't got any plans to move to the United States. But I am interested in the solution, so for the moment I wait until somebody has solved this problem, and publishes it on internet. After that, I will be able to find it throught **Google**.

[Update 2004.07.14]: Google already gives some answers. The best discussion about the answer —**7427466391.com**— is found at the **FogCreek forum**.

First 10-digit prime found in consecutive digits of e

The natural exponential function

$$f(x) = e^x$$



Mathematical Model for Growth

In the formula $\boldsymbol{F} = \boldsymbol{P}(1+\boldsymbol{i})^t$,
the base rate of growth is $(1+\boldsymbol{i})$
and occurs at discrete intervals.

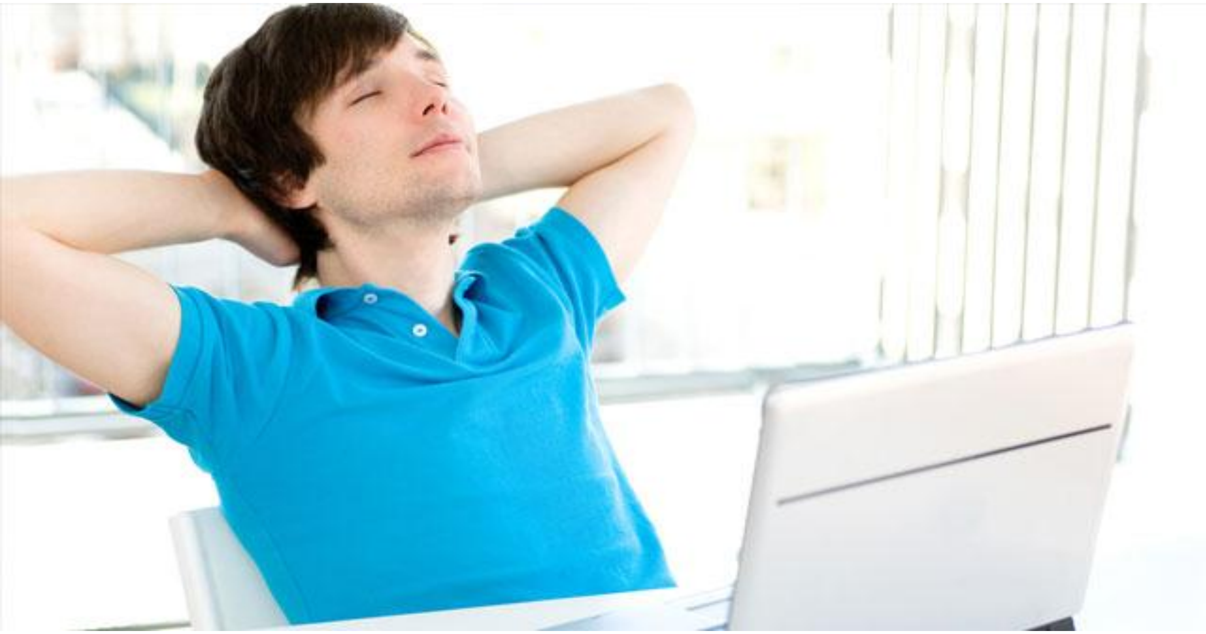
In natural growth, growth occurs continuously and the base rate of growth is $\boldsymbol{e}$.

e

We can approximate discrete growth using e.

- The interest rate was less than 100%?
- The amount invested wasn't €1?

- The investment period was more than 1 year?



Other interest rates, times, principals.........

The final value F, when €1 is invested at 100% continuously compounded interest is

$$F = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

If the interest rate is $i \neq 1$, (still continuous compounding) and $t = 1$ year,

$$F = \lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{n}$$

$$\text{Let } \frac{i}{n} = \frac{1}{x} \Rightarrow n = xi$$

As $n \to \infty,\ x \to \infty$

$$\Rightarrow F = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{xi} = \left(\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)\right)^{i} = e^{i}$$

If the time is t years: $F = \left(e^{i}\right)^{t} = e^{it}$

If the principal is €P: $F = Pe^{it}$

All continuously growing systems are a scaled version of a common rate..

JC & LC Syllabus: All Levels

Students learn about	Students should be able to
2.5 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Leaving Certificate: Strand 3 Number

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to	In addition, students working at HL should be able to
3.1 Number systems	– recognise irrational numbers and appreciate that **R**≠**Q** – revisit the operations of addition, multiplication, subtraction and division in the following domains: • **N** of natural numbers • **Z** of integers • **Q** of rational numbers • **R** of real numbers and represent these numbers on a number line – appreciate that processes can generate sequences of numbers or objects – investigate patterns among these sequences – use patterns to continue the sequence – generate rules/formulae from those patterns	– work with irrational numbers – investigate the operations of addition, multiplication, subtraction and division with complex numbers **C** in rectangular form *a+ib* – illustrate complex numbers on an Argand diagram – interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate – generalise and explain patterns and relationships in algebraic form – recognise whether a sequence is arithmetic, geometric or neither – find the sum to n terms of an arithmetic series – express non-zero positive rational numbers in the form $a \times 10^n$, where $n \in$ **Z** and $1 \leq a < 10$ and perform arithmetic operations on numbers in this form	– geometrically construct $\sqrt{2}$ and $\sqrt{3}$ – prove that $\sqrt{2}$ is not rational – calculate conjugates of sums and products of complex numbers – verify and justify formulae from number patterns – investigate geometric sequences and series – prove by induction • simple identities such as the sum of the first n natural numbers and the sum of a finite geometric series • simple inequalities such as $n! > 2^n$ $2^n > n^2 \ (n \geq 4)$ $(1+x)^n \geq 1+nx \ (x > -1)$

Stock take of problem solving outcomes and methodologies

Leaving Certificate: Strand 3 Number

3.1 Number systems	– recognise irrational numbers and appreciate that **R**≠**Q** – revisit the operations of addition, multiplication, subtraction and division in the following domains: • **N** of natural numbers • **Z** of integers • **Q** of rational numbers • **R** of real numbers and represent these numbers on a number line – appreciate that processes can generate sequences of numbers or objects – investigate patterns among these sequences – use patterns to continue the sequence	– work with irrational numbers – investigate the operations of addition, multiplication, subtraction and division with complex numbers **C** in the form $a+ib$ – illustrate complex numbers on an Argand diagram – interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate – generalise and explain patterns and relationships in algebraic form – recognise whether a sequence is arithmetic, geometric or neither – find the sum to n terms of an arithmetic series – express non-zero positive rational numbers in the form $a \times 10^n$, where $n \in$ **Z** and	– geometrically constru and √3 – calculate conjugates sums and products c complex numbers – verify and justify form from number pattern – investigate geometric sequences and serie – prove by induction • simple identities s as the sum of the natural numbers a sum of a finite gec series • simple inequalities as $n! > 2^n$ $2^n > n^2$ $(n \geq 4)$

Students learn about

3.1 Number systems

Stock take of problem solving outcomes and methodologies

Leaving Certificate: Strand 5 Number

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to	In addition, students working at HL should be able to
5.1 Functions	– recognise that a function assigns a unique output to a given input – graph functions of the form • ax where $a \in \mathbf{Q}$, $x \in \mathbf{R}$ • $ax+b$ where $a,b \in \mathbf{Q}$, $x \in \mathbf{R}$ • ax^2+bx+c where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$ • $a2^x$ where $a \in \mathbf{N}$, $x \in \mathbf{R}$ • $a3^x$ where $a \in \mathbf{N}$, $x \in \mathbf{R}$ – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions – use graphical methods to find approximate solutions to $f(x) = 0$ $f(x) = k$ $f(x) = g(x)$ where $f(x)$ and $g(x)$ are of the above form	– form composite functions – graph functions of the form • ax^3+bx^2+cx+d where $a,b,c,d \in \mathbf{Z}$, $x \in \mathbf{R}$ • ab^x where $a \in \mathbf{N}$, b, $x \in \mathbf{R}$ – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions – use graphical methods to find approximate solutions to $f(x) = 0$ $f(x) = k$ $f(x) = g(x)$ where $f(x)$ and $g(x)$ are of the above form, or where graphs of $f(x)$ and $g(x)$ are provided – investigate the concept of the limit of a function	– recognise surjective, injective and bijective functions – find the inverse of a bijective function – given a graph of a function sketch the graph of its inverse – express quadratic functions in complete square form – use the complete square form of a quadratic function to • find the roots and turning points • sketch the function – graph functions of the form • ax^2+bx+c where $a,b,c \in \mathbf{Q}$, $x \in \mathbf{R}$ • ab^x where $a,b \in \mathbf{R}$ • logarithmic • exponential • trigonometric – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions – informally explore limits and continuity of functions

Leaving Certificate: Strand 5 Functions

$x \in \mathbf{R}$

- $a2^x$ where $a \in \mathbf{N}$, $x \in \mathbf{R}$
- $a3^x$ where $a \in \mathbf{N}$, $x \in \mathbf{R}$

– interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions

– use graphical methods to find approximate solutions to

$f(x) = 0$

$f(x) = k$

$f(x) = g(x)$

where $f(x)$ and $g(x)$ are of the above form

– interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions

– use graphical methods to find approximate solutions to

$f(x) = 0$

$f(x) = k$

$f(x) = g(x)$

where $f(x)$ and $g(x)$ are of the above form or where graphs of $f(x)$ and $g(x)$ are provided

– investigate the concept of the limit of a function

– use the complete square form of a quadratic function to

- find the roots and turning points
- sketch the function

– graph functions of the form

- ax^2+bx+c where $a,b,c \in \mathbf{Q}$, $x \in \mathbf{R}$
- ab^x where $a,b \in \mathbf{R}$
- logarithmic
- exponential
- trigonometric

– interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions