Effective Questions for Introducing Calculus

LINEAR (Slide 31)

Q. Here we have a graph of distance (km) vs time (mins) for a train. How could a student who has completed their JC calculate the speed of the train?
A. (Average) Speed = Distance Travelled / Time Taken

Write the calculation of the speed using the formula on the board.
Q. Can you see where this calculation appears on the graph?
Q. Can you see the distance travelled anywhere on the graph?
A. Choose the pair of points used in the formula calculation and highlight the rise as the distance travelled.

Q. Can you see the time take anywhere in the graph?
A. Highlight the run as the time taken.

Q. Can you identify a mathematical connection here? Is there another way to think of the calculation of the speed of the Train?
A. It's the slope of the graph.

Q. What can we say about the speed of this train? Explain (Think - Pair - Share)
A. It's constant because the slope of a line is constant.

Q. So we've worked out the (Average) speed of the train between two points. Suppose I wanted to find the speed at 4 s exactly. How would I do so?
A. It's the same as the speed between any of the pairs of points. Because the speed is constant.

<u>Key points to have written up on the board by the end of this session:</u>
Slope of Graph is the rate of change of the underlying relationship.
In a linear relationship the rate of change is constant.
In a linear relationship the rate of change at a point is just the rate of change between two points (because the rate of change is constant)

NON-LINEAR

Watch the Usain Bolt video (Slide 32)

Q: At the end of the video ask: So, the fastest ever time for a 100 m race. Just how fast did Usain Bolt run? (What was his speed?)
Calculate the speed on the board using the Speed = Distance Travelled / Time taken formula.

Q: Do you think that he ran at this speed during the entire race?
Q: Why / Why not? Explain your reasoning.

Q: So, if he didn't run at the same speed during the entire race - what does your answer of 10.33 m/s represent?

A: The speed / rate of change over the entire race or the average speed / rate of change during the race.

Refer to learning from board on the train activity and how this calculation compares (in terms of approach)

Here's a graph of the world-record run. (Slide 33)
Q: Does the graph support our hypothesis that the rate of change isn't constant over the entire race?
A: Yes - it's not linear (note it is very close to linear for much of the race)

Sketch the graph on board.
Q: When we looked at our speeding train, we saw that the speed of the train was calculated using (average) speed = distance travelled /time taken but that the speed was also the slope of the graph. We have worked out Usain's average speed using the same formula average speed = distance travelled /time taken. In the case of the world-record run can we also see how to calculate the average speed using the graph.

Q: Can you find the calculation that you have done for 10.45 m/s on the graph? Before answering, draw in the vertical line from the point (9.58,0) to (9.58,100) and horizontal line from (0,0) to (9.58,0) on the board

Q: What have you calculated in the context of the graph with the 10.45? Referring back to the train problem (link slope to rate of change).
A: Slope of this line (join the points (0,0) and (9.58,100).

So, we can see that the slope of a line between two points on a graph gives us the average rate of change between these two points. A line which cuts a curve at two points is called a secant. So, the slope of the secant is the average rate of change between two points on a graph.
The slope of the secant from (0,0) to (9.58,100) is the average speed over the course of the race. (demonstrate on the GeoGebra file)
The slope of the secant from (2.88,20) to (5.xx,50) is the average speed between the 20 and the 50 m mark. (Demonstrate on the GeoGebra file)

Let's look at a different problem. What was Usain's speed at exactly 1 second into the race?

Q: How can we solve this? Did we not look at a similar problem with our speeding train? Will the approach we used there work for us? Why / Why not?

So, in the case where the rate of change is not constant it is not simple to calculate the rate of change at a single point. In fact, it is very difficult to do.
You're in good company in struggling with this problem - this is similar to a problem which occupied some of the greatest mathematicians that ever lived - Isaac Newton and Leibniz.

Q: So, we know the average speed over the entire race - could you suggest how we might get some measure of the speed at the 1 s mark?

We could look at the average speed around the 1 s mark

JUMP TO GEOGEBRA FILE NOW

Click the checkbox “Show Speed” on the GeoGebra file to show the diagram you have sketched on the board

Q: If we pick a point on our graph, say the 1s mark, can we get an estimate of the speed at this point?
On the GeoGebra file, move the points from (0,0) and (9.58,100) stopping at one or two points along the way to ask

Q: How do we improve our estimate of the speed at 1s?
A: Bring the points closer to the 1s mark

Continue to bring the points towards the 1s mark until both points sit on the point at the 1s mark and the slope calculation will disappear.

Q: We can see we have an error here, why did this happen?
A: Division by zero

This is where we introduce the idea of letting the gap between the two points tend towards a limit of zero. An Irish mathematician Berkley identified the use of limits to allow the calculations of Newton to make sense mathematically.

So, to calculate the rate of change at a point we need limits to allow us to predict what happens when two points are brought to a single point.

Geometrically when we do this, we move from calculating the average rate of change between two point using the slope of a secant to calculating the instantaneous rate of change at a single point using the slope of a line which touches the curve at a single point…
The slope of a tangent at a point on a curve gives the instantaneous rate of change at that point.