PDST
Professiona Deve opment Service for Teachers | An tSeirbhís um Fhorbairt Ghairmiúi do Mhúinteoirí

An Roinn Oideachais
Department of Education

Teaching Geometry for Understanding

Resources

https://padlet.com/postprimarymaths/geometryws1

Attendance ✓

Twitter
@ProjectMaths_ie

Newsletter
http://eepurl.com/ghqwLD

General Enquiries:
Postprimarymaths@pdst.ie

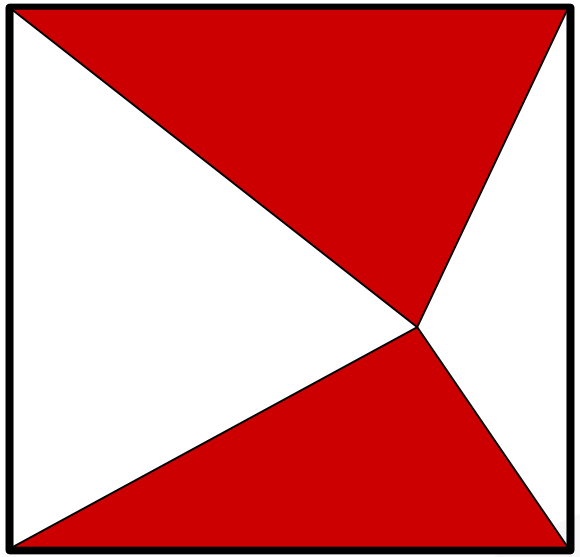
Key Messages

Building on prior knowledge is essential for sense-making in Geometry.

Students progress their geometric thinking based on appropriate social learning experiences.

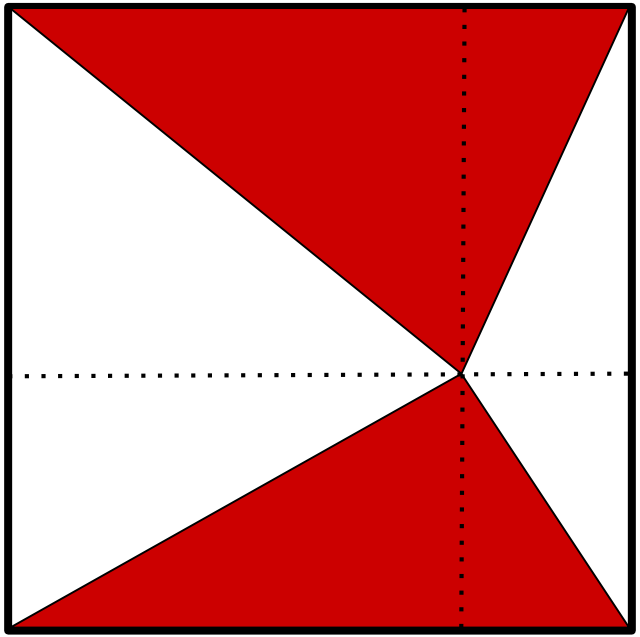
Discussion

A point inside a square is connected to each of it’s four vertices. What fraction of the square is shaded?

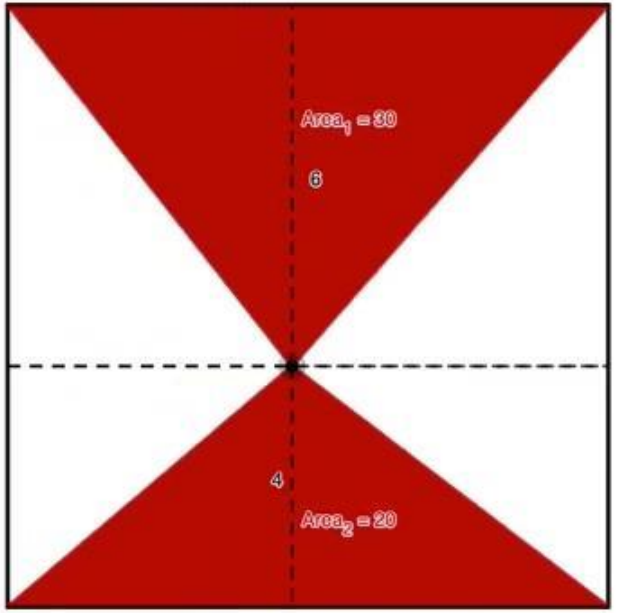


Southall & Pantaloni (2018)

Solutions



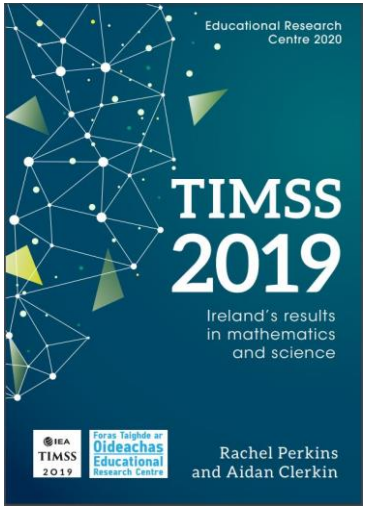
Solutions



Reflection

What challenges for teachers and barriers to students' learning are presented by the teaching of Geometry?

International Research



Number	+17
Data & Chance	+17
Algebra	-18
Geometry	-18

Reasoning -16

PISA

IEA TIMSS & PIRLS International Study Center Lynch School of Education, Boston College

International Research

Table 8.11: Percentages of students taught the TIMSS mathematics topics – Eighth grade

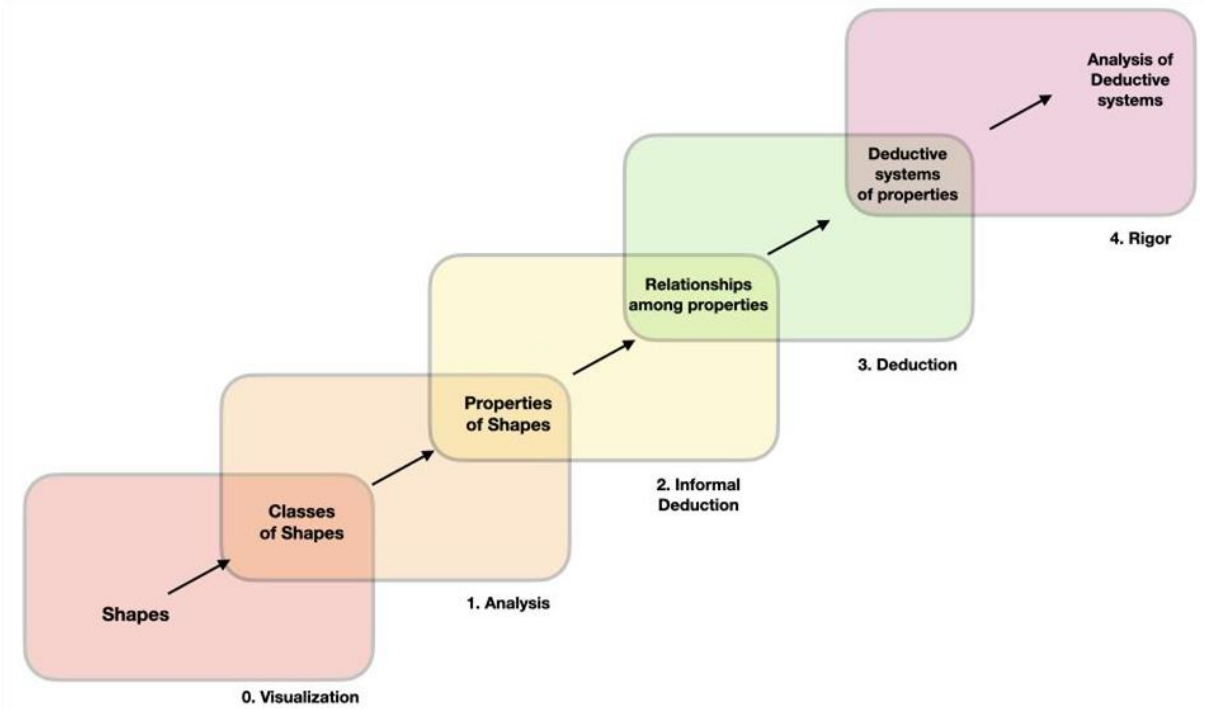
	All mathematics (22 topics) % (*SE*)	Number (3 topics) % (*SE*)	Algebra (7 topics) % (*SE*)	Geometry (6 topics) % (*SE*)	Data & Probability (6 topics) % (*SE*)
Ireland	68 (*1.1*)	99 (*0.3*)	73 (*1.3*)	49 (*2.3*)	66 (*2.2*)
TIMSS	72 (*0.2*)	98 (*0.1*)	68 (*0.2*)	76 (*0.2*)	60 (*0.3*)

Perkins, R. & Clerkin, A. (2020) page 73

PDST
Professiona Deve opment Service for Teachers | An tSeirbhís um Fhorbairt Ghairmiúil do Mhúinteoirí

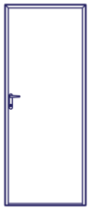
Van Hiele Theory of Geometric Thought

Van Hiele Levels



Van de Wall (2006) page 306

Van Hiele Levels

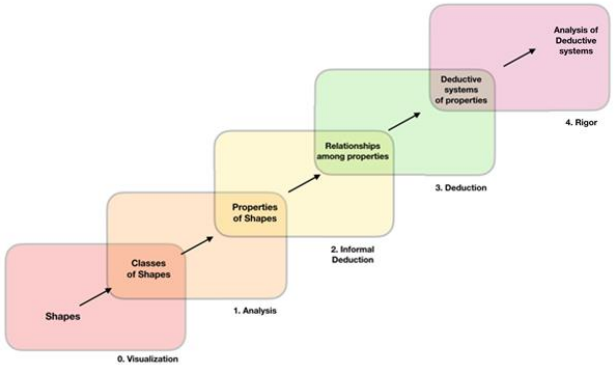
Level 0 (Visualisation)	Distinguish shapes Can not identify properties of shapes	‘A door is a rectangle’. 
Level 1 (Analysis)	Recognise that shapes have different properties.	Identify rectangles regardless of size and orientation. 
Level 2 (Informal Deduction)	See relationships between properties and shapes.	A rectangle is a parallelogram: It has all the properties of a parallelogram as well as having all 90° angles.

Summary

Students progress through levels based on **experiences**.

Students cannot meaningfully engage with content presented at a higher level of thinking without **passing through the previous levels**.

Students can **memorise** but not understand this content from a higher level.



Assessment – Formative Multiple Choice Diagnostic

Fig. 1 Students should be able to answer this question using formal deduction.

Three Properties of a Quadrilateral
Property D: It has diagonals of equal length.
Property S: It is a square.
Property R: It is a rectangle.

Which is true?
a. D implies S, which implies R.
b. D implies R, which implies S.
c. S implies R, which implies D.
d. R implies D, which implies S.
e. R implies S, which implies D.

Source: Usiskin (1992)

Fig. 2 This question can be used as a formative assessment at the integration phase of understanding.

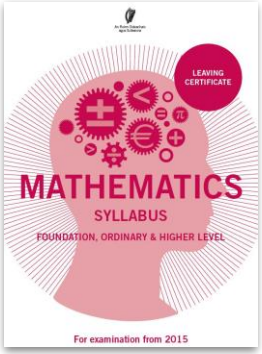
Unit Reflection Question
4. Which of the following statements about similarity are true and which are false?

a. Any two equilateral triangles are similar.
b. Any two rectangles are similar.
c. Any two squares are similar.
d. Any two isosceles triangles are similar.

Source: Lappan et al. 2002, p. 87

Leaving Certificate Syllabus page 23

*Learners should **first encounter** the geometrical results below through **investigation and discovery.** They should come to appreciate that certain features of shapes or diagrams appear to be **independent of the particular examples** chosen.*



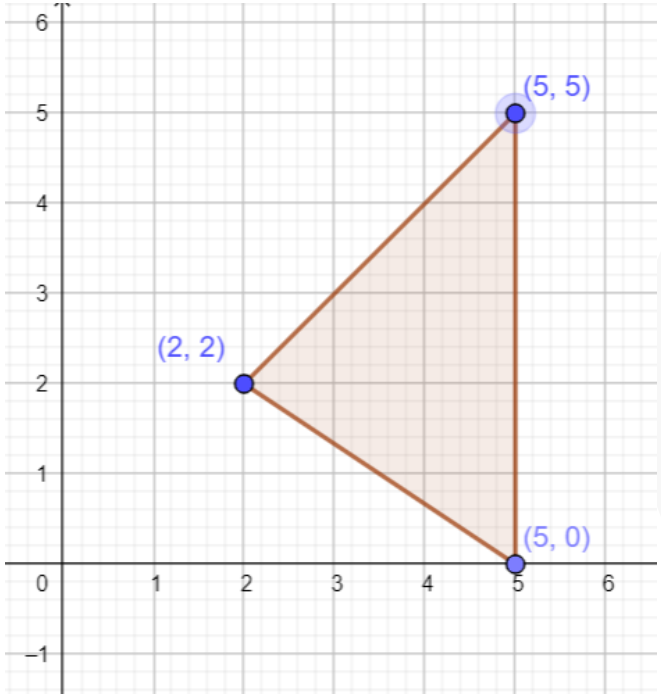
Social Constructivism

- Learning does not occur when the learner passively receives information.
- Learners are the makers of meaning and knowledge, not simply the receivers.
- Learning takes place through language and the dialogue between two or more learners.

Psychology for the Classroom: Constructivism and Social Learning page 52

Activity 2

Find the area of the triangle.



Notes on Activity 2

Questioning

T: Can you explain how you have determined the area of this triangle?
S: Half the base by the perpendicular height
T: Why do you say "½ the base by the height"?
S: Because it is half a rectangle
T: How were you able to get the dimensions of this rectangle?
S: Count the squares.
T: Can you construct this rectangle?
T: What shapes do you see?
S: Right angle triangles and rectangles.
T: What are the properties of these shapes?
S: The right angle triangles are congruent. Thus each triangle is half the area of the rectangle.

It is important to construct a rectangle around the triangle to ensure the students has a visual representation of what they are saying. By drawing the rectangle around the triangle we gain a greater understanding of this geometric shape and its connected properties.

We might ask our students some open questions to stimulate discussion within their group.
Can you make relationships between these properties?
Can you use these properties to deduce the area of the triangle is (½)x base x height

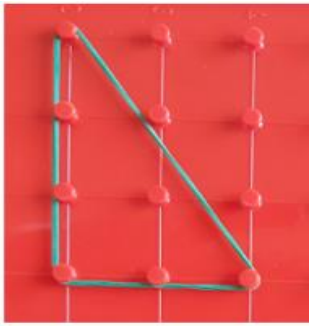
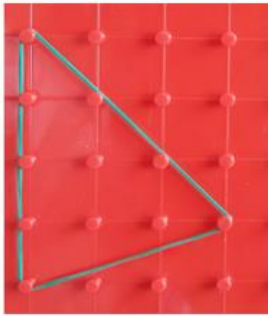
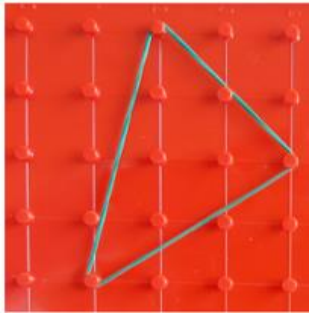
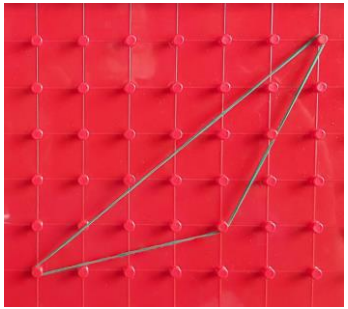
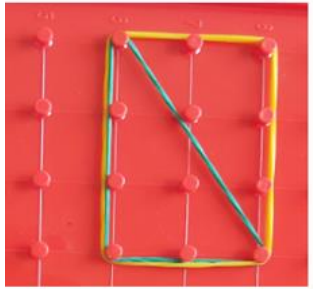
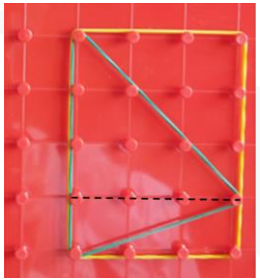
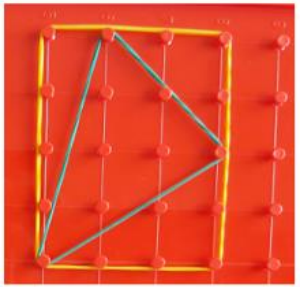
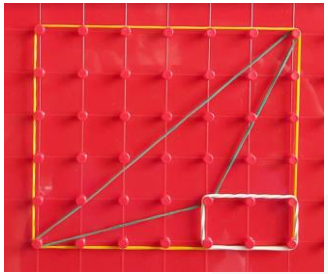
By engaging in effective questioning we can determine if our students are simply using a formula or do they see shapes, properties and can they build relationships.

Area of a triangle

Use the geoboards to construct the following triangles :

1. A right angled triangle.
2. All angles acute and one side vertical.
3. All angles acute and no vertical or horizontal sides.
4. An obtuse angle and no vertical or horizontal sides.

Solutions

Extending the Learning.

The points A(-9, 3), B(-4, 3) and C(-4, 10) are the vertices of the triangle ABC. Find the area of the triangle ABC.

Extension:

Find the area of a triangle with vertices (-3, 4) (4, 2) (6,10).

Class Discussion

Notes on Discussion

Possible Approaches:

Translating one vertex to (0,0) along with the area of a triangle formula.

Possible advantages -

- Demonstrates understanding of transformations. (“recognise images of points and objects under translation” pg 24)
- Connections across the strands (Synthetic, coordinate, algebra).
- Skill - Procedural fluency.

Considerations/disadvantages

- Is the formula applied with purpose and as a result of understanding or have they just dropped it in?
- Not explicitly referenced on syllabus - do we need to use it?
- Will this work for other shapes?

Drawing the diagram and realising that this is a right-angled triangle and therefore it is more trivial to calculate the area.

Possible advantages -

- Problem Solving Strategy - ‘Draw a diagram’
- Simple solution

Considerations/disadvantages

- Will this approach work for triangles that are not right angled?

Creating the rectangle around the triangle.

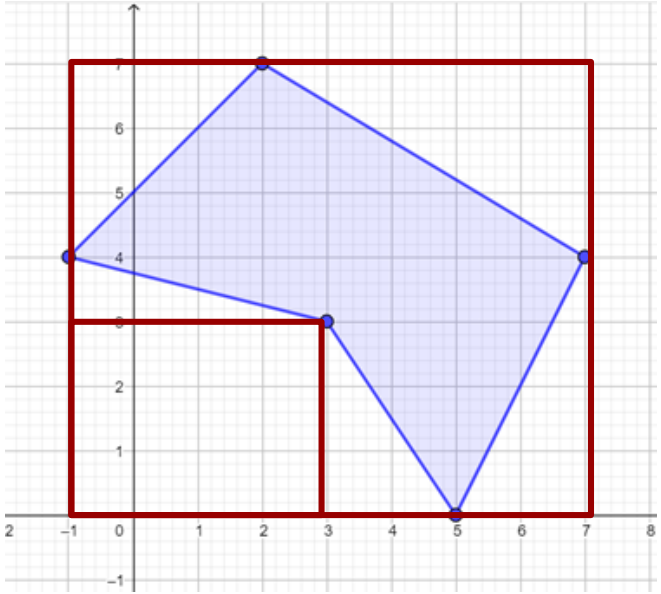
Possible advantages -

- Problem Solving Strategy - ‘simplify the problem’
- Approach can work for multiple settings

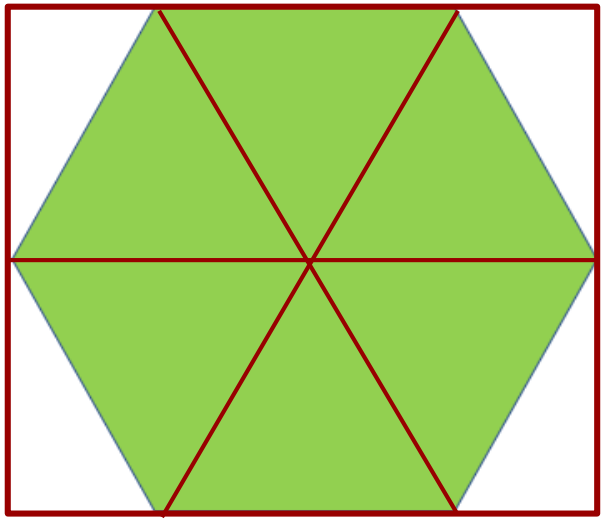
Considerations/disadvantages

- Is this always the most efficient approach?

Area of a Polygon



Area of a Hexagon



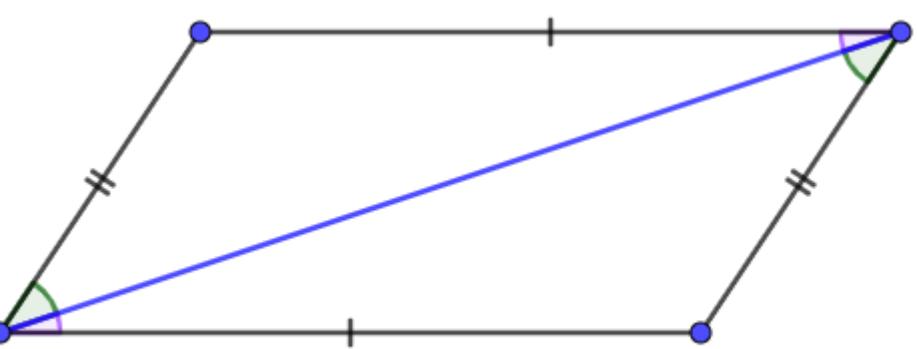
Student Reflection

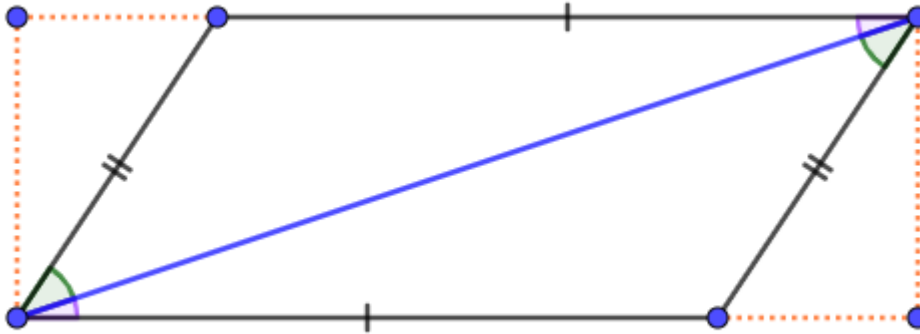
Write down what new learning you encountered during this lesson.

Write down any new terminology that you learnt today.

Provide a detailed description of each new term listed.

Next Steps: Area of a Parallelogram

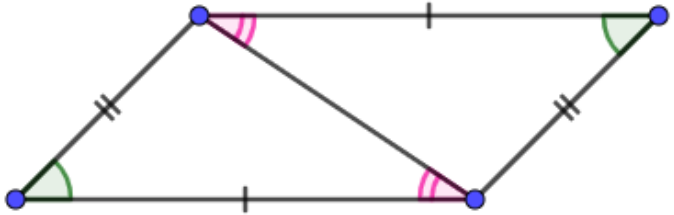




Having explored the approach to finding the area of a polygon and investigated in particular triangles, a natural place to go next might be to extend our approach to quadrilaterals. We mentioned that we want to give our students multiple paths to a solution to developing problem solving skills in order to ensure the mathematics is accessible to students.

If some of our students have not yet reached Level 2 we could draw on our previous investigation where we can use the same strategy to investigate the area of the parallelogram by constructing a rectangle around the parallelogram and then removing the congruent right angle triangles.

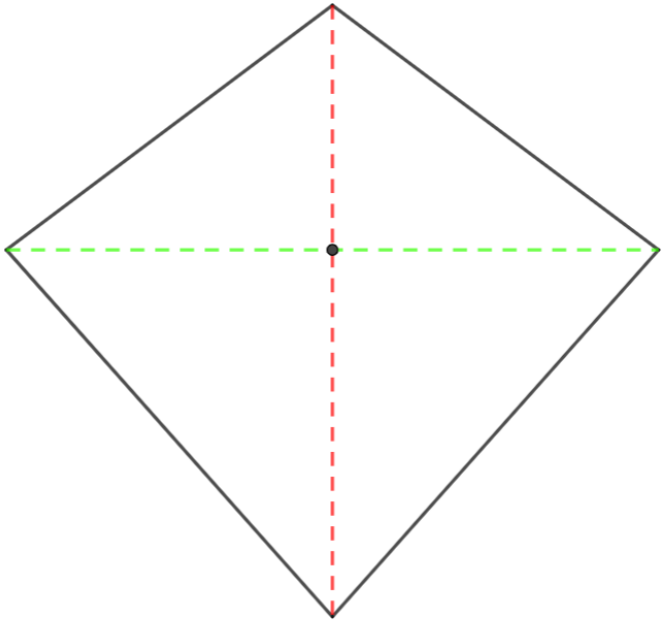
Next Steps: Theorem 17
Diagonal of a parallelogram bisects the area



A natural next step may be to explore the relationships within the quadrilateral. We have considered the area of a quadrilateral in terms of two congruent triangles. We also know (from theorem 9) that in a parallelogram, opposite sides are equal, and opposite angles are equal. We could encourage our students to draw on their prior knowledge of congruent triangles and their understanding of the area of a triangle and with the support of effective questioning our students should be able to informally deduce Theorem 17 " The diagonal of a parallelogram bisects the area".

From this a student operating at Level 2 informal deduction should be able to conclude Theorem 18. The area of a parallelogram is the base by the height.

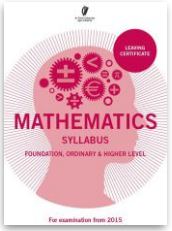
Next Steps: Do the diagonals of any quadrilateral bisect each other?



If our students have a good understanding of the key concepts, there is an opportunity to ask some interesting questions around the nature of quadrilaterals and in particular parallelograms that students can investigate. If a student is operating at Level 2 we could present them with a task to explore a hypothesis. Consider page 71 of the syllabus we could for example ask students to investigate if the diagonals of any quadrilateral bisect each other. (Using informal deduction students should be able to deduce that the converse is not true - consider a kite for example)

Geogebra:
https://www.geogebra.org/classic/udxsefet

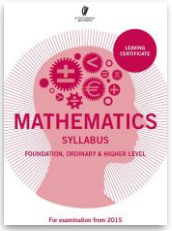
An Integrated Approach



3.4 Length, area and volume	2D shapes and 3D solids, including nets of solids. Using nets to analyse figures and to distinguish between surface area and volume. Problems involving perimeter, surface area and volume. Modelling real-world situations and solving a variety of problems (including multi-step problems) involving surface areas, and volumes of cylinders and rectangular solids. The circle, and develop an understanding of the relationship between its circumference, diameter and π.	– investigate the nets of rectangular solids and cylinders – select and use suitable strategies to find length of the perimeter and the area of the following plane figures: disc, triangle, rectangle, square, and figures made from combinations of these – select and use suitable strategies to estimate the area of a combination of regular and irregular shapes – select and use suitable strategies to find the volume and surface area of rectangular solids, cylinders and spheres – draw and interpret scaled diagrams
		– investigate the nets of prisms, cylinders and cones – solve problems involving the length of the perimeter and the area of plane figures: disc, triangle, rectangle, square, parallelogram, trapezium, sectors of discs, and figures made from combinations of these – solve problems involving surface area and volume of the following solid figures: rectangular block, cylinder, right cone, triangular-based prism (right angle, isosceles and equilateral), sphere, hemisphere, and solids made from combinations of these – use the trapezoidal rule to approximate area

Leaving Certificate Syllabus Page 25, 32, 33

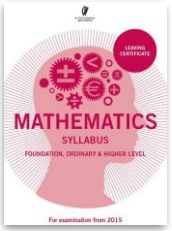
An Integrated Approach



2.4 Transformation geometry, enlargements	Translations, central symmetry, axial symmetry and rotations. Enlargements.	– locate axes of symmetry in simple shapes – recognise images of points and objects under translation, central symmetry, axial symmetry and rotation – investigate enlargements and their effect on area, paying attention to • centre of enlargement • scale factor k where $0<k<1$, $k>1$ $k \in \mathbf{Q}$ – solve problems involving enlargements

Leaving Certificate Syllabus Page 24

An Integrated Approach



2.1 Synthetic geometry	– perform constructions **16-21** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: theorem, proof, axiom, corollary, converse, implies – investigate theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21 and corollary 6 (see Geometry for Post-primary School Mathematics) and use them to solve problems	– perform construction **22** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: is equivalent to, if and only if, proof by contradiction – prove theorems 11,12,13, concerning ratios (see Geometry for Post-primary School Mathematics), which lay the proper foundation for the proof of the theorem of Pythagoras studied at junior cycle
2.2 Co-ordinate geometry	– use slopes to show that two lines are • parallel • perpendicular – recognise the fact that the relationship $ax + by + c = 0$ is linear – solve problems involving slopes of lines – calculate the area of a triangle – recognise that $(x-h)^2 + (y-k)^2 = r^2$ represents the relationship between the x and y co-ordinates of points on a circle with centre (h, k) and radius r – solve problems involving a line and a circle with centre (0, 0)	– solve problems involving • the perpendicular distance from a point to a line • the angle between two lines – divide a line segment internally in a given ratio $m: n$ – recognise that $x^2+y^2+2gx+2fy+c = 0$ represents the relationship between the x and y co-ordinates of points on a circle with centre $(-g,-f)$ and radius r where $r = \sqrt{(g^2+f^2-c)}$ – solve problems involving a line and a circle

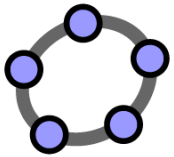
2.3 Trigonometry	– use of the theorem of Pythagoras to solve problems (2D only) – use trigonometry to calculate the area of a triangle – solve problems using the sine and cosine rules (2D) – define sin θ and cos θ for all values of θ – define tan θ – solve problems involving the area of a sector of a circle and the length of an arc – work with trigonometric ratios in surd form	– use trigonometry to solve problems in 3D – graph the trigonometric functions sine, cosine, tangent – graph trigonometric functions of type • $f(\boldsymbol{\theta}) = a+b\text{Sin } c\boldsymbol{\theta}$ • $g(\boldsymbol{\theta}) = a+b\text{Cos } c\boldsymbol{\theta}$ for $a,b,c \in \mathbf{R}$ – solve trigonometric equations such as Sin $n\boldsymbol{\theta}=0$ and Cos $n\boldsymbol{\theta}= ½$ giving all solutions – use the radian measure of angles – derive the trigonometric formulae 1, 2, 3, 4, 5, 6, 7, 9 (see appendix) – apply the trigonometric formulae 1-24 (see appendix)

Leaving Certificate Syllabus Page 25

Activity 3

A Scout Troop have pitched two tents to sleep in and wish to build one fire. Where is the fairest location for the fire?

Justify your answer.



https://www.geogebra.org/classic/u3my5nqb

Notes on Activity 3

T: What do we mean by fair?
S: Same distance from each tent.
T: Is there a word to describe that?
S: Equidistant.

This is the first key concept that we need to ensure our students understand.
Completion of this task will show that the student understands what it means to be equidistant. FOr some students we need to scaffold the discussion to ensure they understand fully the properties of the bisector line.

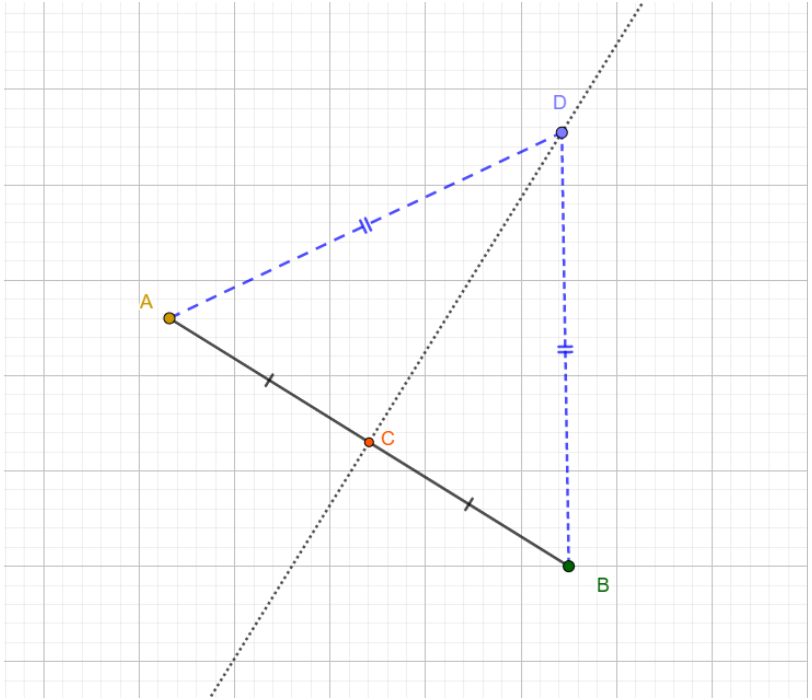
T: Can you find the fairest point between 2 of the tents?
S: The midpoint.

Click on to 16 in construction protocol
T: Why is this fair?
S: It's equidistant from A and B
T: Is there any other point which is equidistant from both tents?
We are prompting here for the students to identify points along the bisector - this shows an understanding of their prior knowledge and the relationships within the bisector line.

Click on to 18
T: How do we know that this point is equidistant?

Click on to the end (22)

Activity 3



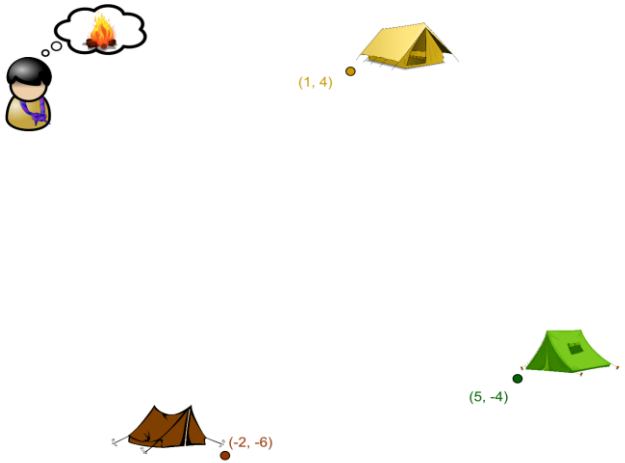
Notes

What does the point C represent?
We would like students to identify that C is the midpoint of the line segment AB
Describe the relationship between the point C and the points A and B?
We want our students to use language such as C is equidistant from A and B.
What does the point D represent?
We would like students to identify that D is any point on the line DC
Describe the relationship between the point D and the points A and B?
We would like students to identify that D is another point that is also equidistant from A and B.
What is the relationship between [AB] and DC?
DC is the perpendicular bisector of the line segment [AB]
What do we know about triangles ADC and DCB?
They are congruent because all the sides and all the angles are the same.

It is important that students use the correct terminology and that their answers are mathematically correct and demonstrate the required understanding. We need to make sure they are comfortable with the prior knowledge before introducing the new concepts.
The key learning for our students that we are assessing here is that the bisector line contains a set of points that are equidistant to the points A and B. They must be able to verbalise this and to understand this key concept in order to proceed with the task.

Extending the Learning

A Scout Troop have pitched three tents to sleep in and wish to build one fire. Where is the fairest location for the fire? Justify your answer.



Support Questions

How do we ensure this is the fairest location, can we verify? (For students who just drop a point) (We can measure the distances to each end point).

What do we mean by fair? (Prompting for 'same distance from each tent' or equidistant)

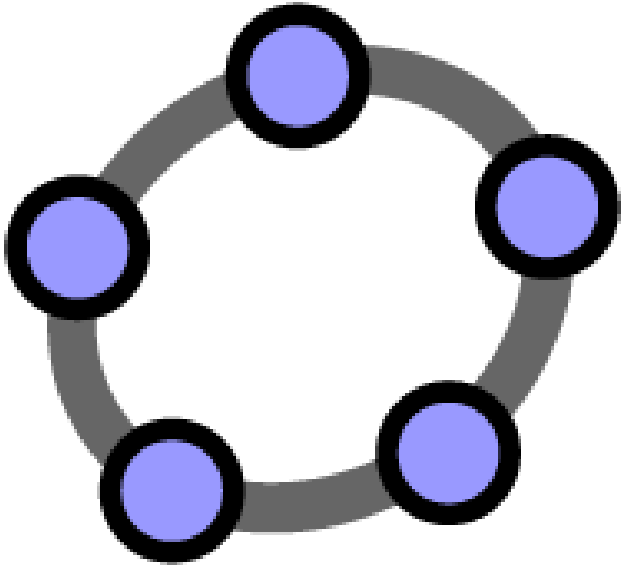
Can you find the fairest point between 2 of the tents? (Requires an understanding of the midpoint)

Is there any other point which is equidistant from both tents? (Prompting for points along the bisector - drawing on prior knowledge)

How can I find the fairest point between 3 tents? (Must apply understanding of bisecting lines)

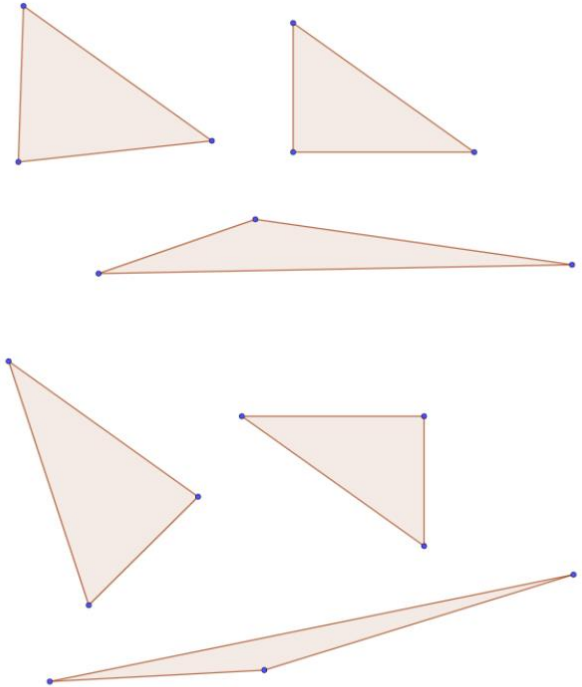
Plot the points on a coordinate grid and investigate the solution.

Class Discussion

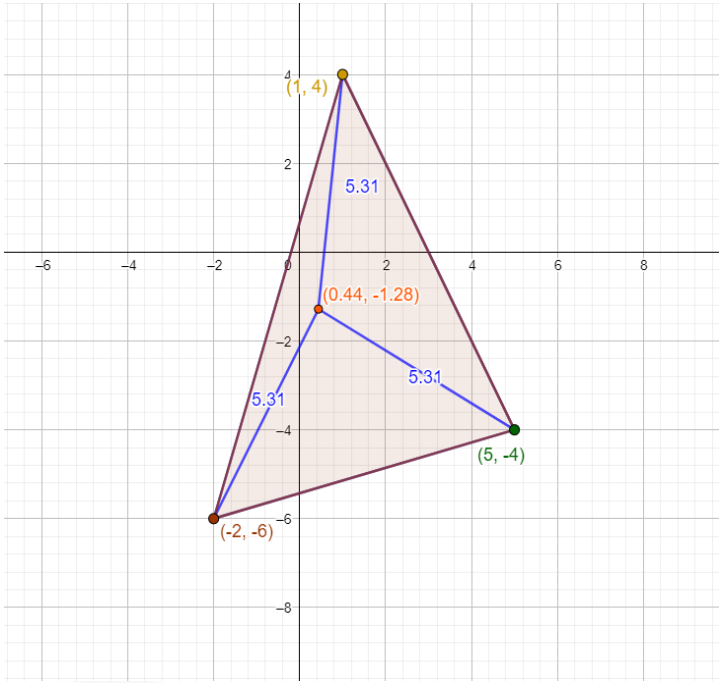


https://www.geogebra.org/classic/xhvcvxsw

Problem: Exploring the Circumcenter



Activity 5: Exploring the Circumcenter

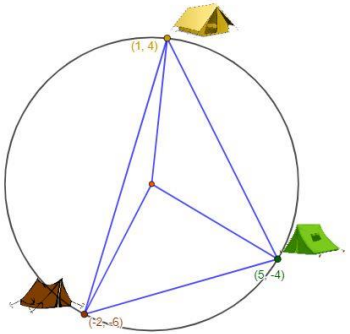


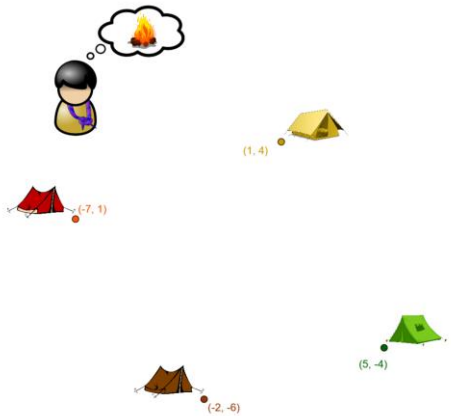
https://www.geogebra.org/classic/s5s7q8ca

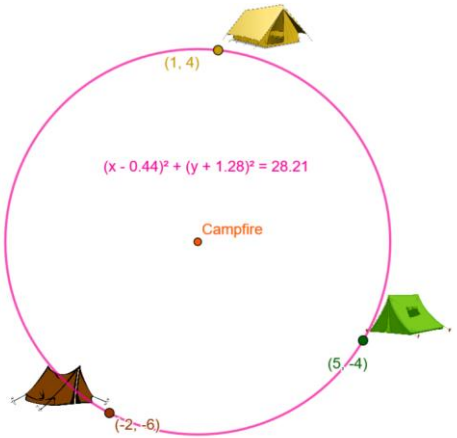
Student Reflection

Write down what new learning you encountered during this lesson.
Write down any new terminology that you learnt today.
Provide a detailed description of the strategies used in this lesson.

Next Steps

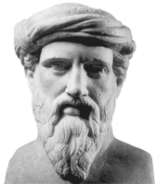


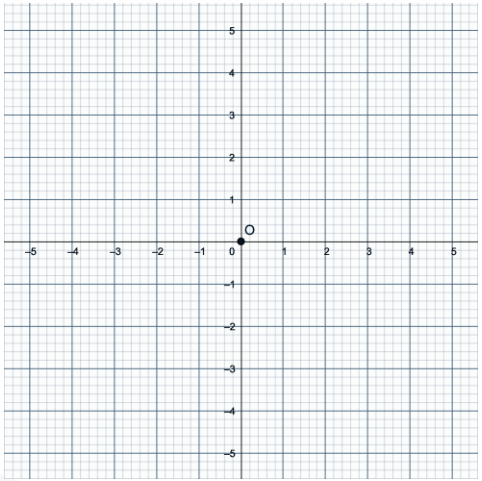


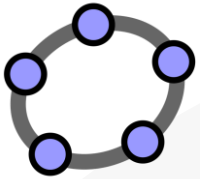


Equation of the Circle

Write down the coordinates of as many points as possible that are 5 units from the origin?







https://www.geogebra.org/classic/b3gqyzsx

An Integrated Approach

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
2.1 Synthetic geometry	– perform constructions **16-21** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: theorem, proof, axiom, corollary, converse, implies – investigate theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21 and corollary 6 (see Geometry for Post-primary School Mathematics) and use them to solve problems	– perform construction **22** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: is equivalent to, if and only if, proof by contradiction – prove theorems 11,12,13, concerning ratios (see Geometry for Post-primary School Mathematics), which lay the proper foundation for the proof of the theorem of Pythagoras studied at junior cycle
2.2 Co-ordinate geometry	– use slopes to show that two lines are • parallel • perpendicular – recognise the fact that the relationship $ax + by + c = 0$ is linear – solve problems involving slopes of lines – calculate the area of a triangle – recognise that $(x-h)^2 + (y-k)^2 = r^2$ represents the relationship between the x and y co-ordinates of points on a circle with centre (h, k) and radius r – solve problems involving a line and a circle with centre (0, 0)	

Key Messages

Building on prior knowledge is essential for sense-making in Geometry.

Students progress their geometric thinking based on appropriate social learning experiences.

Resources

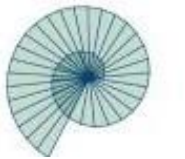
https://padlet.com/postprimarymaths/geometryws1

Twitter
@ProjectMaths_ie

Newsletter
http://eepurl.com/ghqwLD

General Enquiries:
Postprimarymaths@pdst.ie

PDST Post-Primary Maths Team



Team Leader
Stephen Gammell
stephengammell@pdst.ie

Administrator
Gráinne Haughney
grainnehaughneymdt@pdst.ie

Advisors:
advisorname@pdst.ie

Angela Dwane
Arlene Murphy
Darren Murphy
Enda Donnelly
Warren McIntyre
Michael Walsh

References

Breyfogle, L. & Lynch, C. (2010). Mathematics teaching in the Middle school. Vol. 16, No. 4.

DES. (2017). STEM Education Policy Statement.

DES. (2015). Digital Strategy for Schools. Enhancing Teaching, Learning and Assessment.

NCCA. (2016) Junior Cycle Specification.

NCCA. (2013) Junior Certificate Mathematics.

NCCA. (2012) Leaving Certificate Mathematics.

OECD. (2015) PISA 2015 Results (Volume V).

OECD. (2019) Learning for the Future: The performance of 15-Year-olds in Ireland on reading literacy, science and mathematics in PISA 2018.

Pritchard & Woolard (2010). Psychology for the Classroom: Constructivism and Social Learning.

Southall & Pantaloni (2018). More Geometry Snacks. Bitesize problems and multiple ways to solve them.

Van de Wall, J (2006) Elementary and Middle School Mathematics: Teaching Developmentally